



**HAL**  
open science

## Approximate roots of a valuation and the Pierce-Birkhoff Conjecture

François Lucas, James Madden, Daniel Schaub, Mark Spivakovsky

► **To cite this version:**

François Lucas, James Madden, Daniel Schaub, Mark Spivakovsky. Approximate roots of a valuation and the Pierce-Birkhoff Conjecture. 2010. ujm-00461549v1

**HAL Id: ujm-00461549**

**<https://ujm.hal.science/ujm-00461549v1>**

Preprint submitted on 4 Mar 2010 (v1), last revised 9 Feb 2012 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# APPROXIMATE ROOTS OF A VALUATION AND THE PIERCE–BIRKHOFF CONJECTURE

F. Lucas

Département de Mathématiques/CNRS UMR 6093  
Université d'Angers  
2, bd Lavoisier  
49045 Angers Cédex, France

J. Madden

Department of Mathematics  
Louisiana State University at Baton Rouge,  
Baton Rouge, LA, USA

D. Schaub

Département de Mathématiques/CNRS UMR 6093  
Université d'Angers  
2, bd Lavoisier  
49045 Angers Cédex, France

M. Spivakovsky

Inst. de Mathématiques de Toulouse/CNRS 5219  
Université Paul Sabatier  
118, route de Narbonne  
31062 Toulouse cedex 9, France.

## Introduction

All the rings in this paper will be commutative with 1. Let  $R$  be a real closed field. Let  $B = R[x_1, \dots, x_n]$ . If  $A$  is a ring and  $\mathfrak{p}$  a prime ideal of  $A$ ,  $\kappa(\mathfrak{p})$  will denote the residue field of  $\mathfrak{p}$ .

The Pierce–Birkhoff conjecture asserts that any piecewise-polynomial function  $f : R^n \rightarrow R$  can be expressed as a maximum of minima of a finite family of polynomials in  $n$  variables. We start by giving the precise statement of the conjecture as it was first stated by M. Henriksen and J. Isbell in the early nineteen sixties.

**Definition 0.1.1** *A function  $f : R^n \rightarrow R$  is said to be **piecewise polynomial** if  $R^n$  can be covered by a finite collection of closed semi-algebraic sets  $P_i$  such that for each  $i$  there exists a polynomial  $f_i \in B$  satisfying  $f|_{P_i} = f_i|_{P_i}$ .*

Clearly, any piecewise polynomial function is continuous. Piecewise polynomial functions form a ring, containing  $B$ , which is denoted by  $PW(B)$ .

On the other hand, one can consider the (lattice-ordered) ring of all the functions obtained from  $B$  by iterating the operations of sup and inf. Since applying the operations of sup and inf to polynomials produces functions which are piecewise polynomial, this ring is contained in  $PW(B)$

(the latter ring is closed under sup and inf). It is natural to ask whether the two rings coincide. The precise statement of the conjecture is :

**Conjecture 0.1.2 (Pierce-Birkhoff)** *If  $f : R^n \rightarrow R$  is in  $PW(B)$ , then there exists a finite family of polynomials  $g_{ij} \in B$  such that  $f = \sup_i \inf_j (g_{ij})$  (in other words, for all  $x \in R^n$ ,  $f(x) = \sup_i \inf_j (g_{ij}(x))$ ).*

This paper represents the second step of our program for proving the Pierce–Birkhoff conjecture in its full generality. One of the starting points of this program is the abstract formulation of the conjecture in terms of the real spectrum of  $B$  and separating ideals proposed by J. Madden in 1989, [26], which we now recall, together with the relevant definitions. For a general introduction to real spectrum, we refer the reader to [7], Chapter 7, [3], Chapter II or [33], 4.1, page 81 and thereafter ; see also “Bibliographical and historical comments” on p. 109 at the end of that chapter.

Let  $A$  be a ring. A point  $\alpha$  in the real spectrum of  $A$  is, by definition, the data of a prime ideal  $\mathfrak{p}$  of  $A$ , and a total ordering  $\leq$  of the quotient ring  $A/\mathfrak{p}$ , or, equivalently, of the field of fractions of  $A/\mathfrak{p}$ . Another way of defining the point  $\alpha$  is as a homomorphism from  $A$  to a real closed field, where two homomorphisms are identified if they have the same kernel  $\mathfrak{p}$  and induce the same total ordering on  $A/\mathfrak{p}$ .

The ideal  $\mathfrak{p}$  is called the support of  $\alpha$  and denoted by  $\mathfrak{p}_\alpha$ , the quotient ring  $A/\mathfrak{p}_\alpha$  by  $A[\alpha]$ , its field of fractions by  $A(\alpha)$  and the real closure of  $A(\alpha)$  by  $k(\alpha)$ . The total ordering of  $A(\alpha)$  is denoted by  $\leq_\alpha$ . Sometimes we write  $\alpha = (\mathfrak{p}_\alpha, \leq_\alpha)$ .

**Definition 0.1.3** *The real spectrum of  $A$ , denoted by  $\text{Sper } A$ , is the collection of all pairs  $\alpha = (\mathfrak{p}_\alpha, \leq_\alpha)$ , where  $\mathfrak{p}_\alpha$  is a prime ideal of  $A$  and  $\leq_\alpha$  is a total ordering of  $A/\mathfrak{p}_\alpha$ .*

We use the following notation : for an element  $f \in A$ ,  $f(\alpha)$  stands for the natural image of  $f$  in  $A[\alpha]$  and the inequality  $f(\alpha) > 0$  really means  $f(\alpha) >_\alpha 0$ . The real spectrum  $\text{Sper } A$  is endowed with two natural topologies. The first one, called the **spectral (or Harrison) topology**, has basic open sets of the form

$$U(f_1, \dots, f_k) = \{\alpha \mid f_1(\alpha) > 0, \dots, f_k(\alpha) > 0\}$$

with  $f_1, \dots, f_k \in A$ .

The second is the **constructible topology** whose basic open sets are of the form

$$V(f_1, \dots, f_k, g) = \{\alpha \mid f_1(\alpha) > 0, \dots, f_k(\alpha) > 0, g(\alpha) = 0\},$$

where  $f_1, \dots, f_k, g \in A$ . Boolean combinations of sets of the form  $V(f_1, \dots, f_n, g)$  are called **constructible sets** of  $\text{Sper } A$ .

For more information about the real spectrum, see [7]; there is also a brief introduction to the real spectrum and its relevance to the Pierce–Birkhoff conjecture in the Introduction to [21].

**Definition 0.1.4** *Let*

$$f : \text{Sper } A \rightarrow \prod_{\alpha \in \text{Sper } A} A(\alpha)$$

*be a map such that, for each  $\alpha \in \text{Sper } A$ ,  $f(\alpha) \in A(\alpha)$ . We say that  $f$  is **piecewise polynomial** (denoted by  $f \in PW(A)$ ) if there exists a covering of  $\text{Sper } A$  by a finite family  $(S_i)_{i \in I}$  of*

constructible sets, closed in the spectral topology, and a family  $(f_i)_{i \in I}$ ,  $f_i \in A$  such that, for each  $\alpha \in \text{Sper } A$ ,  $f(\alpha) = f_i(\alpha)$ .

We call  $f_i$  a **local representative** of  $f$  at  $\alpha$  and denote it by  $f_\alpha$  ( $f_\alpha$  is, in general, not uniquely determined by  $f$  and  $\alpha$ ; this notation means that one such local representative has been chosen once and for all).

Note that  $PW(A)$  is naturally a lattice ring : it is equipped with the operations of maximum and minimum. Each element of  $A$  defines a piecewise polynomial function. In this way we get a natural injection  $A \subset PW(A)$ .

**Definition 0.1.5** A ring  $A$  is a *Pierce-Birkhoff ring* if, for each  $f \in PW(A)$ , there exist a finite collection of  $f_{ij} \in A$  such that  $f = \sup_i \inf_j f_{ij}$ .

In this paper, Madden reduced the Pierce–Birkhoff conjecture to a purely local statement about separating ideals and the real spectrum. Namely, he gave the following definition :

**Definition 0.1.6** Let  $A$  be a ring. For  $\alpha, \beta \in \text{Sper } A$ , the **separating ideal** of  $\alpha$  and  $\beta$ , denoted by  $\langle \alpha, \beta \rangle$ , is the ideal of  $A$  generated by all the elements  $f \in A$  which change sign between  $\alpha$  and  $\beta$ , that is, all the  $f$  such that  $f(\alpha) \geq 0$  and  $f(\beta) \leq 0$ .

**Definition 0.1.7** A ring  $A$  is **locally Pierce-Birkhoff** at  $\alpha, \beta$  if the following condition holds. Let  $f$  be a piecewise polynomial function, let  $f_\alpha \in A$  be a local representative of  $f$  at  $\alpha$  and  $f_\beta \in A$  a local representative of  $f$  at  $\beta$ . Then  $f_\alpha - f_\beta \in \langle \alpha, \beta \rangle$ .

**Theorem 0.1.8** (Madden) A ring  $A$  is *Pierce-Birkhoff* if and only if it is *locally Pierce-Birkhoff* for all  $\alpha, \beta \in \text{Sper}(A)$ .

Let  $\alpha, \beta$  be points in  $\text{Sper } A$ .

**Conjecture 0.1.9 (local Pierce-Birkhoff conjecture at  $\alpha, \beta$ )** Let  $A$  be a regular ring and  $f$  a piecewise polynomial function. Let  $f_\alpha \in A$  be a local representative of  $f$  at  $\alpha$  and  $f_\beta \in A$  a local representative of  $f$  at  $\beta$ . Then  $f_\alpha - f_\beta \in \langle \alpha, \beta \rangle$ .

There are known counterexamples in the case  $A$  is not regular (eg.  $A = R[x, y]/(y^2 - x^3)$ ) and even with  $A$  normal.

**Remark 0.1.10** Assume that  $\beta$  is a specialization of  $\alpha$ . Then

(1)  $\langle \alpha, \beta \rangle = \mathfrak{p}_\beta$ .

(2)  $f_\alpha - f_\beta \in \mathfrak{p}_\beta$ . Indeed, we may assume that  $f_\alpha \neq f_\beta$ , otherwise there is nothing to prove. Since  $\beta \in \overline{\{\alpha\}}$ ,  $f_\alpha$  is also a local representative of  $f$  at  $\beta$ . Hence  $f_\alpha(\beta) - f_\beta(\beta) = 0$ , so  $f_\alpha - f_\beta \in \mathfrak{p}_\beta$ .

Therefore, to prove that a ring  $A$  is *Pierce-Birkhoff*, it is sufficient to verify the Definition 0.1.7 for all  $\alpha, \beta$  such that neither of  $\alpha, \beta$  is a specialization of the other.

In [21], we introduced

**Conjecture 0.1.11 (the Connectedness conjecture)** Let  $A$  be a regular ring. Let  $\alpha, \beta \in \text{Sper } A$  and let  $g_1, \dots, g_s$  be a finite collection of elements of  $A \setminus \langle \alpha, \beta \rangle$ . Then there exists a connected set  $C \subset \text{Sper } A$  such that  $\alpha, \beta \in C$  and  $C \cap \{g_i = 0\} = \emptyset$  for  $i \in \{1, \dots, s\}$  (in other words,  $\alpha$  and  $\beta$  belong to the same connected component of the set  $\text{Sper } A \setminus \{g_1 \dots g_s = 0\}$ ).

**Definition 0.1.12** A subset  $C$  of  $\text{Sper}(A)$  is said to be **definably connected** if it is not a union of two non-empty disjoint constructible subsets, relatively closed for the spectral topology.

**Conjecture 0.1.13 (Definable connectedness conjecture)** Let  $A$  be a regular ring. Let  $\alpha, \beta \in \text{Sper } A$  and let  $g_1, \dots, g_s$  be a finite collection of elements of  $A$ , not belonging to  $\langle \alpha, \beta \rangle$ . Then there exists a definably connected set  $C \subset \text{Sper } A$  such that  $\alpha, \beta \in C$  and  $C \cap \{g_i = 0\} = \emptyset$  for  $i \in \{1, \dots, s\}$  (in other words,  $\alpha$  and  $\beta$  belong to the same definably connected component of the set  $\text{Sper } A \setminus \{g_1 \dots g_s = 0\}$ ).

In the previous paper [21] we stated the Connectedness conjecture (in the special case  $A = B$ ) and proved that it implies the Pierce–Birkhoff conjecture. Exactly the same proof applies verbatim to show that the Definable Connectedness conjecture implies the Pierce–Birkhoff conjecture for any ring  $A$ .

One advantage of the Connectedness conjecture is that it is a statement about  $A$  (respectively, about the polynomial ring if  $A = B$ ) which makes no mention of piecewise polynomial functions.

Our problem is therefore one of constructing connected subsets of  $\text{Sper } A$  having certain properties.

**Terminology** : If  $A$  is an integral domain, the phrase “valuation of  $A$ ” will mean “a valuation of the field of fractions of  $A$ , non-negative on  $A$ ”. Also, we will sometimes commit the following abuse of notation. Given a ring  $A$ , a prime ideal  $\mathfrak{p} \subset A$ , a valuation  $\nu$  of  $\frac{A}{\mathfrak{p}}$  and an element  $x \in A$ , we will write  $\nu(x)$  instead of  $\nu(x \bmod \mathfrak{p})$ , with the usual convention that  $\nu(0) = \infty$ , which is taken to be greater than any element of the value group.

Given any ordered domain  $D$ , let  $\bar{D}$  denote the convex hull of  $D$  in its field of fractions  $D_{(0)}$  :

$$\bar{D} := \{f \in D_{(0)} \mid d > |f| \text{ for some } d \in D\}.$$

The ring  $\bar{D}$  is a valuation ring, since for any element  $f \in D_{(0)}$ , either  $f \in \bar{D}$  or  $f^{-1} \in \bar{D}$ . We define  $R_\alpha := \overline{A[\alpha]}$ . In this way, to every point  $\alpha \in \text{Sper } A$  we can canonically associate a valuation  $\nu_\alpha$  of  $A(\alpha)$ , determined by the valuation ring  $R_\alpha$ . The maximal ideal of  $R_\alpha$  is  $M_\alpha = \left\{x \in A(\alpha) \mid |x| < \frac{1}{|z|}, \forall z \in A[\alpha] \setminus \{0\}\right\}$ ; its residue field  $k_\alpha$  comes equipped with a total ordering, induced by  $\leq_\alpha$ .

For a ring  $A$ , let  $U(A)$  denote the multiplicative group of units of  $A$ . Recall that  $\Gamma_\alpha \cong \frac{A(\alpha) \setminus \{0\}}{U(R_\alpha)}$

and that the valuation  $\nu_\alpha$  can be identified with the natural homomorphism

$$A(\alpha) \setminus \{0\} \rightarrow \frac{A(\alpha) \setminus \{0\}}{U(R_\alpha)}.$$

By definition, we have a natural ring homomorphism

$$A \rightarrow R_\alpha \tag{1}$$

whose kernel is  $\mathfrak{p}_\alpha$ .

Conversely, the point  $\alpha$  can be reconstructed from the ring  $R_\alpha$  by specifying a certain number of sign conditions (finitely many conditions when  $A$  is noetherian) ([5], [17], [7] 10.1.10, p. 217).

The valuation  $\nu_\alpha$  has the following properties :

- (1)  $\nu_\alpha(A[\alpha]) \geq 0$   
(2) If  $A$  is an  $R$ -algebra then for any positive elements  $y, z \in A(\alpha)$ ,

$$\nu_\alpha(y) < \nu_\alpha(z) \implies y > Nz, \forall N \in R. \quad (2)$$

A  $\nu_\alpha$ -ideal of  $A$  is the preimage in  $A$  of an ideal of  $R_\alpha$ . See [32] or [3], §II.3 for more information on this subject.

As pointed out in [21], the points of  $\text{Sper } A$  admit the following geometric interpretation (see also [10], [15], [32], p. 89 and [34] for the construction and properties of generalized power series rings and fields).

**Definition 0.1.14** *Let  $k$  be a field and  $\Gamma$  an ordered abelian group. The generalized formal power series field  $k((t^\Gamma))$  is the field formed by elements of the form  $\sum_{\gamma \in \Gamma} a_\gamma t^\gamma$ ,  $a_\gamma \in k$  such that the set  $\{\gamma \mid a_\gamma \neq 0\}$  is well ordered.*

The field  $k((t^\Gamma))$  is equipped with the natural  $t$ -adic valuation  $v$  with values in  $\Gamma$ , defined by  $v(f) = \inf\{\gamma \mid a_\gamma \neq 0\}$  for  $f = \sum_{\gamma} a_\gamma t^\gamma \in k((t^\Gamma))$ . The valuation ring of this valuation is the ring  $k[[t^\Gamma]]$  formed by all the elements of  $k((t^\Gamma))$  of the form  $\sum_{\gamma \in \Gamma_+} a_\gamma t^\gamma$ . Specifying a total ordering on  $k$  and  $\dim_{\mathbb{F}_2}(\Gamma/2\Gamma)$  sign conditions defines a total ordering on  $k((t^\Gamma))$ . In this ordering  $|t|$  is smaller than any positive element of  $k$ . For example, if  $t^\gamma > 0$  for all  $\gamma \in \Gamma$  then  $f > 0$  if and only if  $a_{v(f)} > 0$ .

For an ordered field  $k$ , let  $\bar{k}$  denote the real closure of  $k$ . The following result is a variation on a theorem of Kaplansky ([15], [16]) for valued fields equipped with a total ordering.

**Theorem 0.1.15** ([34], p. 62, Satz 21) *Let  $K$  be a real valued field, with residue field  $k$  and value group  $\Gamma$ . There exists an injection  $K \hookrightarrow \bar{k}((t^\Gamma))$  of real valued fields.*

Let  $\alpha \in \text{Sper } A$ . In view of (1) and the Remark above, specifying a point  $\alpha \in \text{Sper } A$  is equivalent to specifying a total order of  $k_\alpha$ , a morphism

$$A[\alpha] \rightarrow \bar{k}_\alpha[[t^{\Gamma_\alpha}]] \quad (3)$$

and  $\dim_{\mathbb{F}_2}(\Gamma_\alpha/2\Gamma_\alpha)$  sign conditions.

We may pass to Zariski spectra to obtain morphisms

$$\text{Spec}(\bar{k}_\alpha[[t^{\Gamma_\alpha}]]) \rightarrow \text{Spec } A[\alpha] \rightarrow \text{Spec } A,$$

induced by the ring homomorphism (3) and the natural surjective homomorphism  $A \rightarrow A[\alpha]$ , respectively.

In particular, if  $\Gamma_\alpha = \mathbb{Z}$ , we obtain a **formal curve** in  $\text{Spec } A$  (an analytic curve if the series are convergent). This motivates the following definition :

**Definition 0.1.16** *Let  $k$  be an ordered field. A  $k$ -**curvette** on  $\text{Sper}(A)$  is a morphism of the form*

$$\alpha : A \rightarrow k[[t^\Gamma]],$$

where  $\Gamma$  is an ordered group. A  $k$ -**semi-curvette** is a  $k$ -curvette  $\alpha$  together with a choice of the sign data  $\text{sgn } x_1, \dots, \text{sgn } x_r$ , where  $x_1, \dots, x_r$  are elements of  $A$  whose  $t$ -adic values induce an  $\mathbb{F}_2$ -basis of  $\Gamma/2\Gamma$ .

We have thus explained how to associate to a point  $\alpha$  of  $\text{Sper } A$  a  $\bar{k}_\alpha$ -semi-curve. Conversely, given an ordered field  $k$ , a  $k$ -semi-curve  $\alpha$  determines a prime ideal  $\mathfrak{p}_\alpha$  (the ideal of all the elements of  $A$  which vanish identically on  $\alpha$ ) and a total ordering on  $A/\mathfrak{p}_\alpha$  induced by the ordering of the ring  $k[[t^\Gamma]]$  of formal power series.

Below, we will often describe points in the real spectrum by specifying the corresponding semi-curves.

Let  $\nu$  be a valuation centered in a regular local ring  $A$  (see §1.1), let  $\Phi = \nu(A \setminus \{0\})$ ;  $\Phi$  is a well-ordered set. For an ordinal  $\lambda < \Phi$ , let  $\gamma_\lambda$  be the element of  $\Phi$  corresponding to  $\lambda$ .

**Definition 0.1.17** *A system of approximate roots of  $\nu$  is a well-ordered set of elements*

$$\mathbf{Q} = \{Q_i\}_{i \in \Lambda} \subset A,$$

*satisfying the following condition : for every  $\nu$ -ideal  $I$  in  $A$ , we have*

$$I = \left\{ \prod_j Q_j^{\gamma_j} \mid \sum_j \gamma_j \nu(Q_j) \geq \nu(I) \right\} A; \quad (4)$$

*furthermore, we require the set  $\mathbf{Q}$  to be minimal in the sense of inclusion among those satisfying (4).*

*A system of approximate roots of  $\nu$  up to  $\gamma_\lambda$  is a well-ordered set of elements of  $A$  satisfying (4) for all the  $\nu$ -ideals  $I$  such that  $\nu(I) < \gamma_\lambda$ .*

The main results of this paper are :

1. Given a regular local ring  $(A, \mathfrak{m})$ , a valuation  $\nu$  centered at  $A$ , as above, and an element  $\gamma_\lambda \in \Phi$  such that the  $\nu$ -ideal determined by  $\gamma_\lambda$  is  $\mathfrak{m}$ -primary, we construct a system of approximate roots up to  $\gamma_\lambda$ .
2. We construct a system of approximate roots for  $A$  and  $\nu$  under the assumption that  $A$  is  $\mathfrak{m}$ -adically complete.
3. In the situation of the Connectedness (or Definable Connectedness) conjecture we describe certain subsets  $C \subset \text{Sper } A$  by explicit formulae in terms of approximate roots ; we conjecture that these sets satisfy the Connectedness (respectively, Definable Connectedness) conjecture.
4. In the special case  $\dim A = 2$ , we use the above results and constructions to prove the Definable Connectedness conjecture (and hence *a fortiori* the Pierce–Birkhoff conjecture). We also prove the Connectedness conjecture in dimension 2, provided the ring  $A$  is excellent.

The paper is organized as follows. Sections 1.1 to 1.6 are purely valuation-theoretic ; sections 1.3 and 1.5 are devoted to the construction of a system of approximate roots.

The approximate roots  $Q_i$  are constructed recursively in  $i$ . Roughly speaking,  $Q_{i+1}$  is the lifting to  $A$  of the minimal polynomial equation satisfied by  $\text{in}_\nu Q_i$  over  $k \left[ \{\text{in}_\nu Q_j\}_{j < i} \right]$  in  $\text{gr}_\nu A$ . In sections 1.1 to 1.6, we prove that such systems of approximate roots exist in two situations : first, for any  $\mathfrak{m}$ -primary  $\nu$ -ideal  $J$  there exists a system of approximate roots up to  $\nu(J)$  ; secondly, there exists a system of approximate roots whenever  $A$  is  $\mathfrak{m}$ -adically complete.

Once these valuation-theoretic tools are developed, we continue with the program announced in [21] for proving the Pierce–Birkhoff conjecture. We place ourselves in the situation of Conjectures 0.1.11 and 0.1.13. In section 1.6, we give an explicit description of a set



$C \subset \text{Sper}A \setminus \{g_1 \dots g_s = 0\}$ , containing  $\alpha$  and  $\beta$ , which we conjecture to be connected. The set  $C$  is described in terms of a finite family of approximate roots, common to the valuations  $\nu_\alpha$  and  $\nu_\beta$ .

Finally, we prove the Definable connectedness conjecture and hence the Pierce-Birkhoff conjecture for an arbitrary regular 2-dimensional local ring  $A$ ; we also prove Conjecture 0.1.11 assuming that  $A$  is excellent which provides a second proof of the Pierce-Birkhoff conjecture in the case of excellent rings.

Our proof is based on Madden's unpublished preprint [27]. As well, we would like to acknowledge a recent paper by S. Wagner [44] which gives a proof of the Definable Connectedness and the Pierce-Birkhoff conjecture in the case of smooth 2-dimensional algebras of finite type over real closed fields. The overall structure of our proof is similar to that of [27] and [44], with the following differences :

1. Here, we have tried to present a proof which should provide a pattern for a general proof of the conjecture, that is, have a hope of generalizing to higher dimensions. In particular, we went to great lengths to phrase everything in terms of approximate roots rather than work directly with connected sets as in [27] and [44].
2. We make no assumptions on the real closedness of the residue field of  $A$  which introduces certain extra complications.
3. Let  $(x, y)$  be a regular system of parameters of  $A$ . Because we work with arbitrary rings, we have to overcome a serious difficulty : proving that the set of points  $\alpha \in \text{Sper} A$ , centered at  $A$  and satisfying  $x(\alpha) > 0$  (respectively,  $x(\alpha) > 0, y(\alpha) > 0$ ) is connected. This is well known for algebras of finite type over a real closed field (see, for example, [7]) but as far as we can tell, for general rings this result is new and non-trivial. Its proof occupies most of section 3.4 and uses a new notion of a graph, associated to a sequence of point blowings-up of a real surface.

## Part 1. Valuations and approximate roots

### 1.1 Generalities on valuations

In this section we review some basic facts of valuation theory.

Let  $A$  be a noetherian ring and  $\nu : A \rightarrow \Gamma \cup \{\infty\}$  a valuation centered at a prime ideal of  $A$ . Let  $\Phi = \nu(A \setminus \{0\}) \subset \Gamma$ .

For each  $\gamma \in \Phi$ , consider the ideals

$$\begin{aligned} P_\gamma &= \{x \in A \mid \nu(x) \geq \gamma\} \\ P_{\gamma+} &= \{x \in A \mid \nu(x) > \gamma\}. \end{aligned} \tag{5}$$

$P_\gamma$  is called the  $\nu$ -**ideal** of  $A$  of value  $\gamma$ .

**Remark 1.1.1** *It is easy to see that, as  $A$  is noetherian,  $\nu(A)$  is well-ordered.*

**Notation.** If  $I$  is an ideal of  $A$  and  $\nu$  a valuation of  $A$ ,  $\nu(I)$  will denote  $\min\{\nu(x) \mid x \in I\}$ .

We now define certain natural graded algebras associated to a valuation. Let  $A$ ,  $\nu$  and  $\Phi$  as above. For  $\gamma \in \Phi$ , let  $P_\gamma$  and  $P_{\gamma+}$  be as in (5). We define

$$\text{gr}_\nu A = \bigoplus_{\gamma \in \Phi} \frac{P_\gamma}{P_{\gamma+}}.$$



The algebra  $\text{gr}_\nu(A)$  is an integral domain. For any element  $f \in A$  with  $\nu(f) = \gamma$ , we may consider the natural image of  $f$  in  $\frac{P_\gamma}{P_{\gamma+}} \subset \text{gr}_\nu(A)$ . This image is a homogeneous element of  $\text{gr}_\nu(A)$  of degree  $\gamma$ , which we denote by  $\text{in}_\nu f$ . The grading induces an obvious valuation on  $\text{gr}_\nu(A)$  with values in  $\Phi$ ; this valuation will be denoted by  $\text{ord}$ .

Next, let  $(A, \mathfrak{m}, k)$  be a local domain and  $\nu$  a valuation with value group  $\Gamma$ , centered at  $A$ . Let  $K$  denote the field of fractions of  $A$  and  $(R_\nu, \mathfrak{m}_\nu, k_\nu)$  the valuation ring of  $\nu$ . For  $\gamma \in \Gamma$ , consider the following  $R_\nu$ -submodules of  $K$  :

$$\begin{aligned} \mathbf{P}_\gamma &= \{x \in R_\nu \mid \nu(x) \geq \gamma\} \\ \mathbf{P}_{\gamma+} &= \{x \in R_\nu \mid \nu(x) > \gamma\}. \end{aligned} \tag{6}$$

We define

$$G_\nu = \bigoplus_{\gamma \in \Phi} \frac{\mathbf{P}_\gamma}{\mathbf{P}_{\gamma+}}.$$

Again, given  $f \in K$ , we may speak of the natural image of  $f$  in  $G_\nu$ , also denoted by  $\text{in}_\nu f$  (since  $\text{gr}_\nu(A)$  is naturally a graded subalgebra of  $G_\nu$ , there is no danger of confusion). Then  $\text{ord}$  is a valuation of the common field of fractions of  $\text{gr}_\nu(A)$  and  $G_\nu$ , with values in  $\Gamma$ .

We end this section with the notion of a *monomial* valuation. Let  $(A, \mathfrak{m}, k)$  be a regular local ring, and  $\mathbf{u} = (u_1, \dots, u_n)$  a regular system of parameters of  $A$ . Let  $\Phi$  be an ordered semigroup and let  $\beta_1, \dots, \beta_n$  be strictly positive elements of  $\Phi$ . Let  $\Phi_*$  denote the ordered semigroup, contained in  $\Phi$ , consisting of all the  $\mathbb{N}_0$ -linear combinations of  $\beta_1, \dots, \beta_n$ . For  $\gamma \in \Phi_*$ , let  $I_\gamma$  denote the ideal of  $A$ , generated by all the monomials  $u^\alpha$  such that  $\sum_{j=1}^n \alpha_j \beta_j \geq \gamma$  (we take  $I_0 = A$ ). Let  $x$  be a non-zero element of  $A$ . Let  $\Phi_x = \{\gamma \in \Phi_* \mid x \in I_\gamma\}$ . Then it is not difficult to prove that the set  $\Phi_x$  contains a maximal element and there exists a unique valuation  $\nu$ , centered at  $\mathfrak{m}$ , such that

$$\nu(u_j) = \beta_j, \quad 1 \leq j \leq n \tag{7}$$

and

$$\nu(x) = \max\{\gamma \in \Phi_x\}, \quad x \in A \setminus \{0\}. \tag{8}$$

This valuation is called the **monomial valuation** of  $A$ , associated to  $\mathbf{u}$  and the  $n$ -tuple  $(\beta_1, \dots, \beta_n)$ . A valuation  $\nu$ , with values in a group  $\Gamma$ , centered in  $\mathfrak{m}$ , is said to be **monomial with respect to  $\mathbf{u}$**  if there exist  $\beta_1, \dots, \beta_n \in \Gamma_+$  such that (8) holds for all  $x \in A \setminus \{0\}$ .

For further results on valuations, see also [43] or [45].

## 1.2 Algebras graded by ordered semigroups

Graded algebras associated to valuations, introduced in the previous section, will play a crucial role in this paper. In this section, we give some basic definitions and prove several easy results about graded algebras. Throughout this paper, a “graded algebra” will mean “an algebra without zero divisors, graded by an ordered semigroup”. As usual, for a graded algebra  $G$ ,  $\text{ord}$  will denote the natural valuation of  $G$ , given by the grading.

**Definition 1.2.1** *Let  $G$  be a graded algebra without zero divisors. The **saturation** of  $G$ , denoted by  $G^*$ , is the graded algebra*

$$G^* = \left\{ \frac{g}{h} \mid g, h \in G, h \text{ homogeneous, } h \neq 0 \right\}.$$

$G$  is said to be **saturated** if  $G = G^*$ .

Of course, we have  $G^* = (G^*)^*$  for any graded algebra  $G$ , so that  $G^*$  is always saturated.

**Example** Let  $\nu$  be a valuation, centered in a local ring  $R$ . Let  $\text{gr}_\nu R$  and  $G_\nu$  be the graded algebras, associated to  $\nu$ , defined in §1.1. We have  $G_\nu = (\text{gr}_\nu R)^*$ ; in particular,  $G_\nu$  is saturated.

**Remark 1.2.2** Let  $G, G'$  be two graded algebras without zero divisors, with  $G \subset G'$ . Let  $x$  be a homogeneous element of  $G'$ , satisfying an algebraic dependence relation

$$a_0 x^\alpha + a_1 x^{\alpha-1} + \cdots + a_\alpha = 0 \quad (9)$$

over  $G$  (here  $a_j \in G$  for  $0 \leq j \leq \alpha$ ). Without loss of generality, we may assume that (9) is homogeneous (that is, the quantity  $j \cdot \text{ord } x + \text{ord } a_j$  is constant for  $0 \leq j \leq \alpha$ ; this is achieved by replacing (9) by the sum of those terms  $a_j x^j$  for which the quantity  $j \cdot \text{ord } x + \text{ord } a_j$  is minimal), and that the integer  $\alpha$  is the smallest possible. Dividing (9) by  $a_0$ , we see that  $x$  satisfies an integral homogeneous relation over  $G^*$  of degree  $\alpha$  and no algebraic relation of degree less than  $\alpha$ . In other words,  $x$  is algebraic over  $G$  if and only if it is integral over  $G^*$ ; the conditions of being “algebraic over  $G^*$ ” and “integral over  $G^*$ ” are one and the same thing.

Let  $G \subset G'$ ,  $x$  be as above and let  $G[x]$  denote the graded subalgebra of  $G'$ , generated by  $x$  over  $G$ . By Remark (1.2.2), we may assume that  $x$  satisfies a homogeneous integral relation

$$x^\alpha + a_1 x^{\alpha-1} + \cdots + a_\alpha = 0 \quad (10)$$

over  $G^*$  and no algebraic relations over  $G^*$  of degree less than  $\alpha$ .

**Lemma 1.2.3** Every element of  $(G[x])^*$  can be written uniquely as a polynomial in  $x$  with coefficients in  $G^*$ , of degree strictly less than  $\alpha$ .

Proof : Let  $y$  be a homogeneous element of  $G[x]$ . Since  $x$  is integral over  $G^*$ , so is  $y$ . Let

$$y^\gamma + b_1 y^{\gamma-1} + \cdots + b_\gamma = 0 \quad (11)$$

with  $b_j \in G^*$ , be a homogeneous integral dependence relation of  $y$  over  $G^*$ , with  $b_\gamma \neq 0$ . By (11),

$$\frac{1}{y} = -\frac{1}{b_\gamma} (y^{\gamma-1} + b_1 y^{\gamma-2} + \cdots + b_{\gamma-1}).$$

Thus, for any  $z \in G[x]$ , we have

$$\frac{z}{y} \in G^*[x]. \quad (12)$$

Since  $y$  was an arbitrary homogeneous element of  $G[x]$ , we have proved that

$$(G[x])^* = G^*[x].$$

Now, for every element  $y \in G^*[x]$  we can add a multiple of (10) to  $y$  so as to express  $y$  as a polynomial in  $x$  of degree less than  $\alpha$ . Moreover, this expression is unique because  $x$  does not satisfy any algebraic relation over  $G^*$  of degree less than  $\alpha$ .  $\square$

**Notation** If  $\Delta \subset \Delta'$  are ordered semigroups and  $\beta$  is an element of  $\Delta'$ ,  $\Delta : \beta$  will denote the positive integer defined by

$$\Delta : \beta = \min\{\alpha \in \mathbb{N} \mid \alpha\beta \in \Delta\}$$

(note that if  $\beta = 0$  then  $\Delta : \beta = 1$ ). If there is no such integer, we put  $\Delta : \beta = \infty$ .

**Lemma 1.2.4** *Let  $G, G'$  and  $x$  be as in Remark 1.2.2. Assume that the degree 0 part of  $G$  contains a field  $k$  and that  $G$  is generated as a  $k$ -algebra by homogeneous elements  $w_1, \dots, w_r$ . Let  $\beta_j = \text{ord } w_j, 1 \leq j \leq r$ , and let  $\Delta$  denote the group  $\Delta = \{\text{ord } y \mid y \in G^*\} \equiv \left\{ \sum_{j=1}^r \alpha_j \beta_j \mid \alpha_j \in \mathbb{Z} \right\}$ .*

*Assume that the following two conditions hold :*

1.  $\Delta : (\text{ord } x) < \infty$
2. Let  $\bar{\alpha} := \Delta : \text{ord } x$ . Let  $\alpha_1, \dots, \alpha_r \in \mathbb{Z}$  be such that

$$\bar{\alpha} \text{ ord } x = \sum_{j=1}^r \alpha_j \beta_j. \quad (13)$$

Let  $y = \prod_{j=1}^r w_j^{\alpha_j}$ . We require that the element

$$z := \frac{x^{\bar{\alpha}}}{y} \in (G')^* \quad (14)$$

be algebraic over  $k$ .

Then  $x$  is integral over  $G^*$ . An integral dependence relation of  $x$  over  $G^*$  can be described as follows. Let  $z$  be as in (14). Let  $Z$  be an independent variable and let

$$\lambda(Z) = Z^d + \sum_{i=0}^{d-1} c_i Z^i = 0 \quad (15)$$

denote the minimal polynomial of  $z$  over  $k$ . Then  $x$  satisfies the integral dependence relation

$$x^{d\bar{\alpha}} + \sum_{i=0}^{d-1} c_i y^{d-i} x^{i\bar{\alpha}} = 0. \quad (16)$$

Conversely, suppose  $x$  is integral over  $G^*$ . Then (1) holds. Suppose, furthermore, that  $\beta_1, \dots, \beta_r$  are  $\mathbb{Z}$ -linearly independent. Then (2) also holds. In this case, (16) is the minimal polynomial of  $x$  over  $G^*$ . In particular, the degree  $\alpha$  of the minimal polynomial of  $x$  over  $G^*$  is given by

$$\alpha = d\bar{\alpha}. \quad (17)$$

Proof : If (1) and (2) hold,  $x$  is integral over  $G^*$  because it satisfies the integral dependence relation (16) (this is verified immediately by substituting (14) for  $Z$  in (15) and multiplying through by  $y^d$ ). In particular, if  $\alpha$  denotes the degree of  $x$  over  $G^*$ , (16) shows that

$$\alpha \leq d\bar{\alpha}. \quad (18)$$

Conversely, suppose  $x$  is integral over  $G^*$ . Then  $x$  satisfies a homogeneous integral relation of the form (10). Since (10) is homogeneous, we have the equality

$$i \text{ ord } x + \text{ord } a_i = j \text{ ord } x + \text{ord } a_j \text{ for some } i, j \text{ such that } 0 \leq i < j \leq \alpha.$$

Hence

$$(j - i) \text{ ord } x = \text{ord } a_i - \text{ord } a_j. \quad (19)$$

Now,  $\frac{a_i}{a_j} \in G^*$  so that

$$\text{ord } a_i - \text{ord } a_j = \text{ord } \frac{a_i}{a_j} \in \Delta. \quad (20)$$

Putting together (19) and (20), we obtain (1) of the Lemma.

Now, assume that  $\beta_1, \dots, \beta_r$  are  $\mathbb{Z}$ -linearly independent. We wish to prove (2). Since  $\beta_1, \dots, \beta_r$  are  $\mathbb{Z}$ -linearly independent, all the monomials  $w_1^{\gamma_1} \cdots w_r^{\gamma_r}$ ,  $\gamma_j \in \mathbb{Z}$ , have different values with respect to  $\text{ord}$ . Since (10) is homogeneous with respect to  $\text{ord}$ , each  $a_i$  must be a *monomial* in the  $w_j$  with (not necessarily positive) integer exponents. Also by the  $\mathbb{Z}$ -linear independence of  $\beta_1, \dots, \beta_r$ , the coefficients  $\alpha_1, \dots, \alpha_r$  in (13) are uniquely determined. Moreover, any relation of the form

$$i \text{ ord } x - \sum_{j=1}^r \alpha'_j \beta_j = 0, \quad i \in \mathbb{N}, \alpha'_1, \dots, \alpha'_r \in \mathbb{Z}. \quad (21)$$

is a positive integer multiple of the relation

$$\bar{\alpha} \text{ ord } x - \sum_{j=1}^r \alpha_j \beta_j = 0. \quad (22)$$

This proves that  $x^i$  may appear in (10) only if  $\bar{\alpha} \mid i$ ; in particular,  $\bar{\alpha} \mid \alpha$ . Let  $d' := \frac{\alpha}{\bar{\alpha}}$ . Let  $0 \leq i < d'$ . To find the coefficient  $a_{i\bar{\alpha}}$  in (10), note that

$$\alpha \text{ ord } x = d' \bar{\alpha} \text{ ord } x = i \bar{\alpha} \text{ ord } x + \text{ord } a_{i\bar{\alpha}},$$

so that

$$(d' - i) \bar{\alpha} \text{ ord } x = \text{ord } a_{i\bar{\alpha}}. \quad (23)$$

Since  $a_{i\bar{\alpha}}$  is a monomial in  $w_1, \dots, w_r$ , (23) gives rise to a  $\mathbb{Z}$ -linear dependence relation of the form (21), which therefore must be equal to (22) multiplied by  $d' - i$ . This determines the monomial  $a_{i\bar{\alpha}}$  uniquely up to multiplication by an element of  $k$ : we must have  $a_{i\bar{\alpha}} = c_i y^{d'-i}$ , where  $c_i \in k$ . Then  $z \equiv \frac{x^{\bar{\alpha}}}{y}$  satisfies the algebraic dependence relation

$$z^{d'} + \sum_{i=0}^{d'-1} c_i z^i = 0. \quad (24)$$

This proves (2) of the Lemma. Now, we have shown that, under the hypothesis of linear independence of the  $\beta_j$ , if  $x$  has degree  $\alpha$  over  $G^*$  then  $\bar{\alpha} \mid \alpha$  and  $z$  satisfies a polynomial of degree  $d' = \frac{\alpha}{\bar{\alpha}}$ . Letting  $d$  denote the degree of  $z$  over  $k$ , as above, we obtain

$$d' \equiv \frac{\alpha}{\bar{\alpha}} \geq d. \quad (25)$$

Combining (25) with (18), we obtain (17); in particular, (16) is the minimal polynomial of  $x$  over  $G$ . This completes the proof of Lemma 1.2.4.  $\square$

**Corollary 1.2.5** *Let  $G$ ,  $w_1, \dots, w_r$  and  $\beta_1, \dots, \beta_r$  be as in Lemma 1.2.4. If  $\beta_1, \dots, \beta_r$  are  $\mathbb{Z}$ -linearly independent in  $\Delta$  then  $w_1, \dots, w_r$  are algebraically independent over  $k$ .*

*Proof*: Induction on  $r$ . For  $r = 1$  there is nothing to prove. For the induction step, assume that the Corollary is true for  $r = i$ . If  $w_{i+1}$  were algebraic over  $k[w_1, \dots, w_i]$ , we would have

$$(\beta_1, \dots, \beta_i) : \beta_{i+1} < \infty \quad (26)$$

by Lemma 1.2.4, applied to the graded algebra  $k[w_1, \dots, w_i]$  and the element  $w_{i+1}$ . Thus (26) contradicts the linear independence of  $\beta_1, \dots, \beta_r$ , and we are done. Alternatively, the Corollary can be proved by observing that, by linear independence of  $\beta_1, \dots, \beta_r$ , all the monomials in  $w$  have different degrees, thus any polynomial in  $w$  over  $k$  contains a unique monomial of smallest degree. Hence such a polynomial cannot vanish.  $\square$

**Definition 1.2.6** Let  $G$  be a graded algebra and  $x_\Lambda := \{x_\lambda\}_{\lambda \in \Lambda}$  a collection of elements of  $G$ . Let  $k$  be a field, contained in the degree 0 part of  $G$ . Let  $k[x_\Lambda]$  denote the  $k$ -subalgebra of  $G$ , generated by  $x_\Lambda$ . We say that  $x_\Lambda$  **rationaly generate**  $G$  if  $G^* = k[x_\Lambda]^*$ .

The following result is an immediate consequence of definitions :

**Proposition 1.2.7** Let  $G_\nu$  be the graded algebra associated to a valuation  $\nu : K \rightarrow \Gamma$ , as above. Consider a sum of the form  $y = \sum_{i=1}^s y_i$ , with  $y_i \in K$ . Let  $\beta = \min_{1 \leq i \leq s} \nu(y_i)$  and

$$S = \{i \in \{1, \dots, n\} \mid \nu(y_i) = \beta\}.$$

The following two conditions are equivalent :

- (1)  $\nu(y) = \beta$
- (2)  $\sum_{i \in S} \text{in}_\nu y_i \neq 0$ .

### 1.3 Approximate roots up to $\nu(J)$ for an $\mathfrak{m}$ -primary ideal $J$

Let  $A$  be a regular local ring of dimension  $n$ ,  $\mathfrak{m}$  its maximal ideal,  $k = \frac{A}{\mathfrak{m}}$ ,  $\mathbf{u} = (u_1, \dots, u_n)$  a regular system of parameters and

$$\nu : A \setminus \{0\} \rightarrow \Gamma$$

a valuation, centered in  $\mathfrak{m}$  (this means  $\nu(\mathfrak{m}) > 0$ ).

Let  $\mathbf{1} = \nu(\mathfrak{m}) = \min\{\gamma \in \Phi \mid \gamma > 0\}$  and  $\Phi_1 = \{\gamma \in \Phi \mid \exists a \in \mathbb{N}; \gamma < a \cdot \mathbf{1}\}$ . For the sake of simplicity, we will write  $a$  instead of  $a \cdot \mathbf{1}$ . We shall study the structure of  $\nu$ -ideals  $P_\gamma$  where  $\gamma \in \Phi$ .

If  $\nu$  were monomial with respect to  $\mathbf{u}$  then  $\text{in}_\nu u_1, \dots, \text{in}_\nu u_n$  would generate  $\text{gr}_\nu A$  as a  $k$ -algebra. We are interested in analyzing valuations which are not necessarily monomial. We fix an  $\mathfrak{m}$ -primary valuation ideal  $J$ . The purpose of sections 1.3 and 1.4 is to construct a system of approximate roots up to  $\nu(J)$ , that is, a finite sequence of elements  $\mathbf{Q} = \{Q_i\}_{i \in \Lambda}$  of  $A$  such that for every  $\nu$ -ideal  $I$  in  $A$  containing  $J$  we have

$$I = \left\{ \prod_j Q_j^{\gamma_j} \mid \sum_j \gamma_j \nu(Q_j) \geq \nu(I) \right\} A \quad (27)$$

(in particular, the images  $\text{in}_\nu Q_i$  of the  $Q_i$  in  $G_\nu$  generate  $G_\nu$  as a  $k$ -algebra up to degree  $\nu(J)$ ). In this construction, each  $Q_{i+1}$  will be described by an explicit formula (given later in this section) in terms of  $Q_1, \dots, Q_i$ .

The earliest precursor of approximate roots appears in a series of papers by Saunders MacLane and O.F.G. Schilling [23], [24] and [25]. In dimension 2, they were defined globally in  $k[x, y]$  by S. Abhyankar and T. T. Moh ([1], [2]) and locally by M. Lejeune-Jalabert [20]. See

also the papers [18] and [19] by T. C. Kuo, [12] by R. Goldin and B. Teissier and [36] by M. Spivakovsky, [11] by F.J. Herrera Govantes, M.A. Olalla Acosta, M. Spivakovsky, [39]-[42] by Michel Vaquié. We also refer the reader to the paper [38] by B. Teissier for a different approach to the theory of approximate roots in higher dimensions.

Let  $k = \frac{A}{\mathfrak{m}} = \frac{A}{\mathfrak{m}_\nu \cap A}$  be the residue field of  $A$ . Fix an isomorphism  $\frac{A}{J} \cong \frac{k[u_1, \dots, u_n]}{J_0}$ , where  $J_0$  is an ideal of  $k[u_1, \dots, u_n]$ . In this way, we will view  $k$  as a subring of  $A/J$ .

We fix, once and for all, a section  $k \rightarrow A$  of the natural map  $A \rightarrow k$  which composed with the natural map  $A \rightarrow \frac{A}{J}$  maps  $k$  isomorphically onto its image in  $\frac{A}{J}$ . The image of  $k$  in  $A$  will be denoted by  $\mathbf{k}$ .

According to Definition 0.1.17, we are looking for a finite set of elements  $\mathbf{Q} = \{Q_i\}_{i \in \Lambda}$ ,  $Q_i \in A$  satisfying (27).

**Remark 1.3.1** *This means, in particular, that the initial forms  $in_\nu(Q_1), in_\nu(Q_2), \dots$  generate  $gr_\nu(A)$ , up to degree  $\nu(J)$ . In other words, we want to build  $\mathbf{Q}$  such that, for  $f \in A$ , we have  $in_\nu(f) \in k[in_\nu \mathbf{Q}]$  provided  $\nu(f) < \nu(J)$ .*

Since  $J$  is an  $\mathfrak{m}$ -primary ideal, there are only finitely many elements of  $\Phi$  less than  $\nu(J)$ . We proceed by induction on the finite set  $\{\gamma \in \Phi \mid \gamma < \nu(J)\}$ .

**Definition 1.3.2** *Let  $E = \{Q \in A\}$  be a set of elements of  $A$ . A generalized monomial  $\mathbf{Q}^\alpha$  in  $E$  is a formal expression*

$$\mathbf{Q}^\alpha = \prod_{Q \in E} Q^{\alpha_Q}$$

where  $\alpha_Q \in \mathbb{N}$  and  $\alpha_Q = 0$  for all  $Q$  outside of a finite subset of  $E$ .

We order the set of generalized monomials by the lexicographical order of the pairs  $(\nu(\mathbf{Q}^\alpha), \alpha)$ .

The semigroup  $\Phi$  is well ordered. For a natural number  $\lambda$ ,  $\gamma_\lambda$  will denote the  $\lambda$ -th element of  $\Phi$ .

We start by choosing a coordinate system adapted to the situation.

**Definition 1.3.3** *Take  $j \in \{2, \dots, n\}$ . We say that  $u_j$  is  $(\nu, J)$ -prepared if either  $u_j \in J$  or there does not exist  $f \in A$  such that*

$$in_\nu u_j = in_\nu f \quad \text{and} \quad (28)$$

$$f \pmod J \in \frac{k[u_1, \dots, u_{j-1}]}{k[u_1, \dots, u_{j-1}] \cap J_0}. \quad (29)$$

*The coordinate system  $\mathbf{u} = \{u_1, \dots, u_n\}$  is  $(\nu, J)$ -prepared if  $u_j$  is  $(\nu, J)$ -prepared for all  $j \in \{2, \dots, n\}$ .*

**Proposition 1.3.4** *There exists a  $(\nu, J)$ -prepared coordinate system.*

*Proof :* We construct a  $(\nu, J)$ -prepared coordinate system recursively in  $j$ . Assume that  $u_1, \dots, u_{j-1}$  are already  $(\nu, J)$ -prepared, but  $u_j$  is not. Take  $f \in A$  satisfying (28) and (29).

Let  $\tilde{u}_j = u_j - f$ ; then  $\nu(\tilde{u}_j) > \nu(u_j)$ .

Since there are only finitely many elements of  $\Phi$  less than  $\nu(J)$ , after finitely many repetitions of the above procedure, we may assume that  $u_j$  is  $(\nu, J)$ -prepared. This completes the proof by induction on  $j$ .  $\square$

We construct, recursively in  $\lambda$ , two finite ordered sets  $\Lambda(\gamma_\lambda)$  and  $\Theta(\gamma_\lambda)$ . At each step we define additional finite ordered sets  $\mathcal{V}(\gamma_\lambda) \subset \Psi(\gamma_\lambda) \subset \Lambda(\gamma_\lambda)$ . A typical element of each of those sets will have the form  $(Q, \text{Ex}(Q))$  where  $Q \in A$  and  $\text{Ex}(Q)$  is a sum of monomials in  $\Lambda(\gamma_\lambda)$ , written in the increasing order according to the order we defined on monomials. Given an element  $(Q, \text{Ex}(Q)) \in \Lambda(\gamma_\lambda) \cup \Theta(\gamma_\lambda)$ ,  $Q$  is called an *approximate root* and  $\text{Ex}(Q)$  is called the *expression* of  $Q$ .

For a natural number  $\ell$ ,  $\gamma_\ell \leq \nu(J)$ , and for  $(Q, \text{Ex}(Q)) \in \Lambda(\gamma_\ell) \cup \Theta(\gamma_\ell)$ , let  $\text{In } Q$  denote the smallest monomial of  $\text{Ex}(Q)$ . Let  $\text{In}(\ell) = \{ \alpha \in \mathbb{N}^{\Psi(\gamma_\ell)} \mid \exists Q \in \Lambda(\gamma_\ell) \text{ such that } \mathbf{Q}^\alpha = \text{In } Q \}$ .

**Theorem 1.3.5** *For a natural number  $\lambda$ ,  $\gamma_\lambda \leq \nu(J)$ , there exist finite ordered sets*

$$\mathcal{V}(\gamma_\lambda) \subset \Psi(\gamma_\lambda) \subset \Lambda(\gamma_\lambda)$$

and  $\Theta(\gamma_\lambda)$  consisting of elements  $(Q, \text{Ex}(Q))$ , with  $Q \in A$  and  $\text{Ex}(Q)$  is a sum of monomials in  $\mathcal{V}(\gamma_\lambda)$ , increasing with respect to the given order on monomials, and having the following properties :

$$\nu(Q) < \gamma_\lambda \text{ whenever } (Q, \text{Ex}(Q)) \in \Lambda(\gamma_\lambda) \quad (30)$$

$$\nu(Q) \geq \gamma_\lambda \text{ whenever } (Q, \text{Ex}(Q)) \in \Theta(\gamma_\lambda). \quad (31)$$

Moreover, for any  $(Q, \text{Ex}(Q)) \in \Lambda(\gamma_\lambda)$ , any monomial  $\mathbf{Q}^\alpha$  appearing in  $\text{Ex}(Q)$  is a monomial in  $\Psi(\gamma_{\lambda-2})$  provided  $Q \notin \{u_1, \dots, u_n\}$ . For any  $(Q, \text{Ex}(Q)) \in \Theta(\gamma_\lambda)$ , any monomial  $\mathbf{Q}^\alpha$  appearing in  $\text{Ex}(Q)$  is a monomial in  $\Psi(\gamma_{\lambda-1})$  provided  $Q \notin \{u_1, \dots, u_n\}$ . An element

$$(Q, \text{Ex}(Q)) \in \Psi(\gamma_\lambda) \cup \Theta(\gamma_\lambda)$$

is completely determined by  $\text{In } Q$ .

Proof : We proceed by induction on  $\lambda$ .

First define  $\Psi(\mathbf{1}) = \Lambda(\mathbf{1}) = \emptyset$  and  $\Theta(\mathbf{1}) = \{(u_1, u_1), \dots, (u_n, u_n)\}$  where we assume

$$\nu(u_1) \leq \nu(u_2) \leq \dots \leq \nu(u_n).$$

Let  $\lambda > 0$  be a natural number such that  $\gamma_\lambda \leq \nu(J)$ . Assume that for each  $\ell < \lambda$  we have constructed sets  $\mathcal{V}(\gamma_\ell) \subset \Psi(\gamma_\ell) \subset \Lambda(\gamma_\ell)$  and  $\Theta(\gamma_\ell)$  having the properties required in the theorem.

Let

$$\Lambda(\gamma_\lambda) = \Lambda(\gamma_{\lambda-1}) \cup \{(Q, \text{Ex}(Q)) \in \Theta(\gamma_{\lambda-1}) \mid \nu(Q) < \gamma_\lambda\}. \quad (32)$$

**Definition 1.3.6** *An element  $(Q, \text{Ex}(Q)) \in \Lambda(\gamma_\lambda)$  is an **inessential predecessor** of an approximate root  $(Q', \text{Ex}(Q')) \in \Lambda(\gamma_\lambda)$  if  $\text{Ex}(Q') = \text{Ex}(Q) + \sum_{\alpha} c_{\alpha} \mathbf{Q}^{\alpha}$ , where  $c_{\alpha} \in k$  and the  $\mathbf{Q}^{\alpha}$  are monomials in  $\Lambda(\gamma_\lambda)$ .*

*An element  $(Q, \text{Ex}(Q)) \in \Lambda(\gamma_\lambda)$  is said to be **essential at the level**  $\gamma_\lambda$  if  $Q$  is not an inessential predecessor of an element of  $\Lambda(\gamma_\lambda)$ .*



**Remark 1.3.7** *Note that, because the coordinate system is prepared,  $u_1, \dots, u_n$  are always essential.*

Let  $\Psi(\gamma_\lambda)$  be the subset of  $\Lambda(\gamma_\lambda)$  consisting of all the essential roots at the level  $\gamma_\lambda$ . Let  $\mathcal{V}(\gamma_\lambda)$  be the subset of  $\Psi(\gamma_\lambda)$  consisting of all  $(Q, \text{Ex}(Q))$  such that  $\text{in}_\nu(Q)$  does not belong to the  $k$ -vector space of  $G = \text{gr}_\nu(A)$  generated by the set  $\{\text{in}_\nu(\mathbf{Q}^\gamma)\}$  where  $\mathbf{Q}^\gamma$  runs over the set of all the generalized monomials on roots preceding  $Q$  in the above ordering.

We extend the total ordering from  $\Lambda(\gamma_{\lambda-1})$  to  $\Lambda(\gamma_\lambda)$  by postulating that  $\Lambda(\gamma_{\lambda-1})$  is the initial segment of  $\Lambda(\gamma_\lambda)$ .

For a natural number  $\ell$ , let  $E(\ell) = \text{In}(\ell) + \mathbb{N}^{\mathcal{V}(\gamma_\ell)} \subset \mathbb{N}^{\mathcal{V}(\gamma_\ell)}$ .

Now consider the ordered set  $\{\mathbf{Q}^{\alpha_1}, \dots, \mathbf{Q}^{\alpha_s}\}$  of monomials

$$\mathbf{Q}^\alpha = \prod Q^{\alpha_Q}, \quad (Q, \text{Ex}(Q)) \in \mathcal{V}(\gamma_\lambda) \cup \{(Q, \text{Ex}(Q)) \in \Theta(\gamma_{\lambda-1}) \mid \nu(Q) = \gamma_\lambda\}, \quad \alpha \notin E(\lambda)$$

of value  $\gamma_\lambda$ .

Let  $i_1 = \max \left\{ i \in \{1, \dots, s\} \mid \text{in}_\nu(\mathbf{Q}^{\alpha_i}) \in \sum_{j=i+1}^s k \text{in}_\nu(\mathbf{Q}^{\alpha_j}) \right\}$  and consider the unique relation  $\text{in}_\nu(\mathbf{Q}^{\alpha_{i_1}}) - \sum_{j=i_1+1}^s \overline{c_{1j}} \text{in}_\nu(\mathbf{Q}^{\alpha_j}) = 0$ . Let  $P_1 = \mathbf{Q}^{\alpha_{i_1}} - \sum_{j=i_1+1}^s c_{1j} \mathbf{Q}^{\alpha_j}$  where  $c_{1j} \in \mathbf{k}$  is the image of  $\overline{c_{1j}}$  under the chosen section  $k \rightarrow A$ .

Let  $i_2 = \max \left\{ i \in \{1, \dots, i_1 - 1\} \mid \text{in}_\nu(\mathbf{Q}^{\alpha_i}) \in \sum_{j=i+1}^s k \text{in}_\nu(\mathbf{Q}^{\alpha_j}) \right\}$  and, as before, consider the unique  $P_2 = \mathbf{Q}^{\alpha_{i_2}} - \sum_{\substack{j=i_2+1 \\ j \neq i_1}}^s c_{2j} \mathbf{Q}^{\alpha_j}$  such that the vector  $(\alpha_j)_{j=i_1+1, \dots, s}$ ,  $\overline{c_{2j}} \neq 0$ , is minimal in the lexicographical order. We continue in this way and define  $P_3, \dots, P_t$ .

Let

$$\Theta(\gamma_\lambda) = \{(Q, \text{Ex}(Q)) \in \Theta(\gamma_{\lambda-1}) \mid \nu(Q) \geq \gamma_\lambda\} \cup \{(P_1, \text{Ex}(P_1)), \dots, (P_t, \text{Ex}(P_t))\} \quad (33)$$

where

$$\text{Ex}(P_j) = \text{Ex}(Q) - \sum_k c_{jk} \mathbf{Q}^{\alpha_k} \quad (34)$$

if  $\mathbf{Q}^{\alpha_{i_j}} = Q$  with  $(Q, \text{Ex}(Q)) \in \mathcal{V}(\gamma_\lambda) \cup \{(Q, \text{Ex}(Q)) \in \Theta(\gamma_{\lambda-1}) \mid \nu(Q) = \gamma_\lambda\}$  and

$$\text{Ex}(P_j) = \mathbf{Q}^{\alpha_{i_j}} - \sum_k c_{jk} \mathbf{Q}^{\alpha_k} \quad (35)$$

otherwise.

**Remark 1.3.8** *Suppose given two approximate roots  $Q_1$  and  $Q_2$  such that*

$$\text{In}(Q_1) = \text{In}(Q_2) = \mathbf{Q}^\alpha$$

*and suppose that  $Q_1$  appears before  $Q_2$  in the process of construction of the approximate roots described above. Because of the uniqueness of the construction of the  $P_i$ 's above, we have*

$$\nu(Q_2) > \nu(Q_1).$$

Now, if  $\nu(\mathbf{Q}^\alpha) = \gamma_\ell$ , then  $\alpha \in E(\ell)$ , so the only way the monomial  $\mathbf{Q}^\alpha$  can appear as an initial form of  $Q_2$  is when  $P_k = Q' + \sum c_j \mathbf{Q}^{\alpha_j}$  where  $\text{In}(Q') = \mathbf{Q}^\alpha$  and then  $\nu(Q') < \nu(Q_2)$ . Then, either  $\nu(Q') = \nu(Q_1)$  and so  $Q' = Q_1$  because of the uniqueness in the construction process, or  $\nu(Q') > \nu(Q_1)$ , but we conclude by descending induction that  $Q_2 = Q_1 + \sum c_j \mathbf{Q}^{\alpha_j}$  and  $\text{Ex}(Q_2) = \text{Ex}(Q_1) + \sum c_j \mathbf{Q}^{\alpha_j}$ .

So finally, the expression of an approximate root has the form

$$\text{Ex}(Q) = \mathbf{Q}^{\alpha_{i_j}} + \sum_k a_k \mathbf{Q}^{\alpha_k} \quad (36)$$

the sum being written in the increasing order of the monomials.

**Remark 1.3.9** *This construction is very similar to finding a basis of the space of relations by row reduction.*

**Remark 1.3.10** *We just showed that there is a one to one correspondence  $Q \leftrightarrow \text{In}(Q)$  between the approximate roots  $Q \in \Psi(\gamma_\ell)$  and the set of monomials which are the first term of the expression  $\text{Ex}(Q)$  of such an approximate root  $Q$ . Let us denote by  $\mathbf{M}(\ell)$  the set of those monomials.*

The last part of the theorem holds by construction.  $\square$

#### 1.4 Standard form up to $\nu(J)$

Consider the integer  $\lambda$  such that  $\gamma_\lambda = \nu(J)$ . Assume that the system of coordinates  $u$  of  $A$  is  $(\nu, J)$ -prepared.

**Definition 1.4.1** *A monomial in  $\Psi(\gamma_\lambda) \cup \Theta(\gamma_\lambda)$  is called **standard with respect to  $\lambda$**  if all the approximate roots appearing in it belong to  $\mathcal{V}(\gamma_\lambda)$  and it is not divisible by any  $\text{In}(Q)$  where  $Q$  is an approximate root in  $(\Psi(\gamma_\lambda) \cup \Theta(\gamma_\lambda)) \setminus \{(u_1, u_1), \dots, (u_n, u_n)\}$ .*

**Definition 1.4.2** *Let  $f \in A$  and let  $\ell$  be a positive integer,  $\ell \leq \lambda$ . An expression of the form*

$$f = \sum c_\alpha \mathbf{Q}^\alpha,$$

*where the  $\mathbf{Q}^\alpha$  are monomials in  $\Psi(\gamma_\lambda) \cup \Theta(\gamma_\lambda)$ , written in the increasing order, is a **standard form of level  $\gamma_\ell$**  with respect to  $\lambda$  if for all  $\gamma' < \gamma_\ell$  and for all  $\alpha$  such that  $\nu(\mathbf{Q}^\alpha) = \gamma'$  and  $c_\alpha \neq 0$ ,  $\mathbf{Q}^\alpha$  is a standard monomial with respect to  $\lambda$ .*

We now construct, by induction on  $\ell$ , a standard form of  $f$  of level  $\gamma_\ell$ . We will write this standard form as

$$f = f_\ell + \sum c_\alpha \mathbf{Q}^\alpha$$

where, for all  $\alpha$ ,  $\mathbf{Q}^\alpha$  is a generalized monomial in  $\Psi(\gamma_\lambda) \cup \Theta(\gamma_\lambda)$ ,  $\nu(\mathbf{Q}^\alpha) \geq \gamma_\ell$  and  $f_\ell$  is a sum of standard monomials in  $\mathcal{V}(\gamma_\lambda)$  of value strictly less than  $\gamma_\ell$ .

To start the induction, let  $f_0 = 0$ . The *standard form of  $f$  of level 0 with respect to  $\lambda$*  will be its expansion  $f = f_0 + \sum c_\alpha \mathbf{u}^\alpha$  as a formal power series in the  $u_i$ , with the monomials written in the increasing order according to the monomial order defined above,.

Let  $\ell$  be a natural number,  $\ell < \lambda$ . Let us define  $f_{\ell+1}$  and the standard form of  $f$  of level  $\gamma_{\ell+1}$  as follows. Assume we already have an expression  $f = f_\ell + \sum c_\alpha \mathbf{Q}^\alpha$  with  $\nu(\mathbf{Q}^\alpha) \geq \gamma_\ell$ , for all  $\alpha$ , and the value of any monomial of  $f_\ell$  is strictly less than  $\gamma_\ell$ .

Take the homogeneous part of  $\sum c_\alpha \mathbf{Q}^\alpha$  of value  $\gamma_\ell$ , with the monomials arranged in the increasing order, and consider the first monomial  $\mathbf{Q}^\alpha$  which is not standard. Since  $\mathbf{Q}^\alpha$  is not standard, one of the two following conditions holds :

1. There exists an approximate root  $Q \in (\Psi(\gamma_\lambda) \cup \Theta(\gamma_\lambda)) \setminus \{(u_1, u_1), \dots, (u_n, u_n)\}$  such that  $In(Q)$  divides  $\mathbf{Q}^\alpha$ . Write  $Q = In(Q) + \sum c_\beta \mathbf{Q}^\beta$  and replace  $In(Q)$  by  $Q - \sum c_\beta \mathbf{Q}^\beta$  in  $\mathbf{Q}^\alpha$ .
2. There exists  $Q \in \Psi(\gamma_\lambda) \setminus \mathcal{V}(\gamma_\lambda)$  which divides  $\mathbf{Q}^\alpha$ . Since  $Q \notin \mathcal{V}(\gamma_\lambda)$ , there exists

$$Q' \in \Psi(\gamma_\lambda) \cup \Theta(\gamma_\lambda)$$

of the form  $Q' = Q + \sum_\delta d_\delta \mathbf{Q}^\delta$  where  $\mathbf{Q}^\delta$  are monomials in  $\mathcal{V}(\gamma_\lambda)$  of value greater than or equal to  $\gamma_\ell$ . Replace  $Q$  by  $Q' - \sum_\delta d_\delta \mathbf{Q}^\delta$ .

In both cases, those changes introduce new monomials, but either they are of value strictly greater than  $\gamma_\ell$  or they are of value exactly  $\gamma_\ell$  but greater than  $\mathbf{Q}^\alpha$  in the monomial ordering. We repeat this procedure as many times as we can. After a finite number of steps, no more changes are available at level  $\gamma_{\ell+1}$ . Then, let  $f_{\ell+1} = f_\ell + \sum d_\beta \mathbf{Q}^\beta$  with  $\nu(\mathbf{Q}^\beta) = \gamma_\ell$ , so that  $f = f_{\ell+1} + \sum c_\alpha \mathbf{Q}^\alpha$  where  $\nu(\mathbf{Q}^\alpha) > \gamma_\ell$ .

The expression thus constructed satisfies the definition of standard form of level  $\gamma_{\ell+1}$  because all the non-standard monomials  $\mathbf{Q}^\alpha$  of value less than or equal to  $\gamma_\ell$  have been eliminated.

**Proposition 1.4.3** *Let*

$$f = f_\ell + \sum c_\alpha \mathbf{Q}^\alpha$$

*be a standard form of  $f$  of level  $\gamma_\ell$  and  $\gamma < \gamma_\ell$  an element of  $\Phi$ . Then  $\sum_{\nu(\mathbf{Q}^\beta)=\gamma} c_\beta \mathbf{Q}^\beta \notin P_{\gamma+}$ .*

*Proof :* We give a proof by contradiction. Suppose there exists a relation of the form

$$\sum_{\nu(\mathbf{Q}^\beta)=\gamma} c_\beta \mathbf{Q}^\beta \in P_{\gamma+}. \quad (37)$$

Let  $\mathbf{Q}^\alpha$  be the smallest monomial on the left hand side of (37). By construction of approximate roots, there exists a finite collection  $Q_1, \dots, Q_s \in \Lambda(\gamma+) \cup \theta(\gamma+)$  and generalized monomials  $\mathbf{Q}^{\omega_1}, \dots, \mathbf{Q}^{\omega_s}$  such that  $\sum_{i=1}^s Q_i \mathbf{Q}^{\omega_i} = \sum_{\nu(\mathbf{Q}^\beta)=\gamma} c_\beta \mathbf{Q}^\beta$ .

There exists  $i \in \{1, \dots, s\}$  such that one of the two conditions holds : either  $\mathbf{Q}^\alpha = \mathbf{Q}^{\omega_i} \cdot In(Q_i)$  or  $Q_i = Q'_i + \sum_\epsilon b_\epsilon \mathbf{Q}^\epsilon$ ,  $Q'_i \in \Lambda(\gamma+) \setminus \Psi(\gamma+)$ . In either case, the monomial  $\mathbf{Q}^\alpha$  is not standard, which gives the desired contradiction.  $\square$

For each  $\ell$ , the part  $f_\ell$  of a standard form of  $f$  of level  $\gamma_\ell$  is uniquely determined. This is a straightforward consequence of the Proposition.

As a consequence of Proposition 1.4.3, note that if  $\gamma_\ell > \nu(f)$  then  $\nu(f)$  equals the smallest value of a monomial appearing in the standard form of  $f$  of level  $\gamma_\ell$ .

**Theorem 1.4.4** (1) *Take  $\gamma \in \Phi$ ,  $\gamma < \gamma_\lambda$ . Then  $\frac{P_\gamma}{P_{\gamma+}}$  is generated as a  $k$ -vector space by  $\{in_\nu \mathbf{Q}^\beta\}$*

*where  $\mathbf{Q}^\beta$  runs over the set of all the standard monomials with respect to  $\lambda$ , satisfying  $\nu(\mathbf{Q}^\beta) = \gamma$ .*

(2) *The part of the graded  $k$ -algebra  $gr_\nu(A)$  of degree strictly less than  $\gamma_\lambda$  is generated by the initial forms of the approximate roots of  $\mathcal{V}(\gamma_\lambda)$ .*

Proof : Take an element  $\gamma \in \Phi$ ,  $\gamma < \gamma_\lambda$ . Let  $h \in P_\gamma/P_{\gamma+}$  be a homogeneous element of degree  $\gamma$  of  $\text{gr}_\nu(A)$  and let  $f \in P_\gamma$  be such that  $\text{in}_\nu(f) = h$ . Let  $\sum c_\beta \mathbf{Q}^\beta$  denote the homogeneous part of least value of a standard form of  $f$  of level  $\gamma_\lambda$ . Then the initial form of  $f$  is  $\sum \text{in}_\nu(c_\beta \mathbf{Q}^\beta)$ .  $\square$

**The Alvis–Johnston–Madden example.** Let  $\alpha$  be the point of  $\text{Sper}(R[x, y, z])$  given by the curve  $x(t) = t^6$ ,  $y(t) = t^{10} + ut^{11}$ ,  $z(t) = t^{14} + t^{15}$  where  $u$  is some fixed element of  $R$  with  $u > 2$ . Let  $J$  be a  $\nu_\alpha$ -ideal of value greater than or equal to 37.

The calculation of the first few approximate roots gives

$$Q_1 = x, \quad (38)$$

$$Q_2 = y, \quad (39)$$

$$Q_3 = z, \quad (40)$$

$$Q_4 = y^2 - xz = (2u - 1)t^{21} + u^2t^{22}, \quad \nu(Q_4) = 21 \quad (41)$$

$$Q_5 = yz - x^4 = (u + 1)t^{25} + ut^{26}, \quad \nu(Q_5) = 25 \quad (42)$$

$$Q_6 = z^2 - x^3y = (2 - u)t^{29} + t^{30}, \quad \nu(Q_6) = 29 \quad (43)$$

$$Q_7^{(31)} = yQ_4 - \alpha(u)xQ_5, \quad \alpha(u) = (2u - 1)/(u + 1), \quad \nu(Q_7^{(31)}) = 32 \quad (44)$$

$$Q_7^{(32)} = yQ_4 - \alpha(u)xQ_5 - \beta(u)x^3z, \quad \nu(Q_7^{(32)}) = 33 \quad (45)$$

$$Q_7^{(33)} = yQ_4 - \alpha(u)xQ_5 - \beta(u)x^3z - \gamma(u)x^2Q_4 \quad (46)$$

$$Q_7^{(34)} = yQ_4 - \alpha(u)xQ_5 - \beta(u)x^3z - \gamma(u)x^2Q_4 - \delta(u)x^4y \quad (47)$$

$$Q_7^{(35)} = yQ_4 - \alpha(u)xQ_5 - \beta(u)x^3z - \gamma(u)x^2Q_4 - \delta(u)x^4y - \epsilon(u)xQ_6 \quad (48)$$

$$Q_8^{(35)} = zQ_4 + \zeta(u)xQ_6 \quad (49)$$

$$Q_9^{(35)} = yQ_5 + \eta(u)xQ_6, \quad (50)$$

where  $\beta(u)$ ,  $\gamma(u)$ ,  $\delta(u)$ ,  $\epsilon(u)$ ,  $\zeta(u)$ ,  $\eta(u)$  are functions of  $u$  which can be calculated explicitly.

The elements listed above belong to  $\Lambda(37)$ ; we chose to index them as  $Q_i^{(j)}$ . In this notation, the approximate root  $Q_i^{(j)}$  is an inessential predecessor of  $Q_i^{(j+1)}$  whenever  $Q_i^{(j+1)}$  is defined.

We also note the relation  $xQ_6 - yQ_5 + zQ_4 = 0$ , which is the simplest example of a syzygy, an important phenomenon, responsible for much of the difficulty of the Pierce–Birkhoff conjecture.

In the same vein, we can describe the standard form of different levels of an element of  $A$ , say for instance,

$$f = x^3 + y^3 + z^3 \quad (51)$$

(which is a standard form of level 0). For  $\gamma \leq 30$ , the standard form of  $f$  of level  $\gamma$  is given by (51). Then, as  $y^2 \in E(21)$ , we replace  $y^3$  by  $y(Q_4 + xz)$  to obtain

$$f = x^3 + yQ_4 + xyz + z^3. \quad (52)$$

Since  $yz \in E(25)$ , we replace  $xyz$  in (52) by  $xQ_5 + x^5$ , to obtain the standard form of level 31 :

$$f = x^3 + x^5 + yQ_4 + xQ_5 + z^3 \quad (53)$$

(the monomials being written in the order of increasing values 18, 30, 31, 31, 42). Next, we replace  $yQ_4$  by  $\alpha(u)xQ_5$  in (53), so the standard form of levels 32, 33, 34 and 35 is given by

$$f = x^3 + x^5 + (1 + \alpha(u))xQ_5 + \beta(u)x^3z + \gamma(u)x^2Q_4 + \delta(u)x^4y + Q_7^{(34)} + z^3,$$

and so on ...

Let  $\ell$  be an integer such that  $\gamma_\ell \leq \gamma_\lambda$ . Let  $X = X_{\mathcal{V}(\gamma_\ell)}$  be a set of independent variables, indexed by  $\mathcal{V}(\gamma_\ell)$ , and consider the graded  $k$ -algebra  $k[X_{\mathcal{V}(\gamma_\ell)}]$ , where we define  $\deg X_j = \nu(Q_j)$ . Let  $P$  denote the homogeneous monomial ideal of  $k[X_{\mathcal{V}(\gamma_\ell)}]$  generated by all the monomials in  $X_{\mathcal{V}(\gamma_\ell)}$  of degree greater than or equal to  $\gamma_\ell$ . We have a natural map

$$\begin{aligned} \phi_\ell : \frac{k[X_{\mathcal{V}(\gamma_\ell)}]}{P} &\rightarrow \frac{\text{gr}_\nu A}{P_{\gamma_\ell}} \\ X_j &\mapsto \text{in}_\nu Q_j. \end{aligned}$$

Now, for  $\ell = 0$ , let  $I_0 = (0)$ . For  $\ell > 0$ , let  $I_\ell$  denote the ideal of  $\frac{k[X_{\mathcal{V}(\gamma_\ell)}]}{P}$  generated by all the homogeneous polynomials of the form

$$X^{\alpha_0} + \lambda_1 X^{\alpha_1} + \lambda_2 X^{\alpha_2} + \cdots + \lambda_{b_0} X^{\alpha_{b_0}} \quad (54)$$

where  $\mathbf{Q}^{\alpha_0} + \lambda_1 \mathbf{Q}^{\alpha_1} + \lambda_2 \mathbf{Q}^{\alpha_2} + \cdots + \lambda_{b_0} \mathbf{Q}^{\alpha_{b_0}}$  is the homogeneous part of least degree of  $Ex(Q)$  for an approximate root  $Q \in \mathcal{V}(\gamma_\ell) \cup \Theta(\gamma_\ell)$ .

**Corollary 1.4.5** *We have  $\ker \phi_\ell = I_\ell$ .*

Proof : The inclusion  $I_\ell \subset \ker \phi_\ell$  is immediate. To prove the opposite inclusion, we argue by contradiction. Take a homogeneous element

$$h = a_{\lambda_1} X^{\lambda_1} + a_{\lambda_2} X^{\lambda_2} + \cdots + a_{\lambda_s} X^{\lambda_s} \in \ker(\phi_\ell) \setminus I_\ell \quad (55)$$

of degree  $b$ ,  $b < \gamma_\ell$ , such that  $\lambda_1$  is lexicographically smallest among all the elements  $h \in \ker(\phi_\ell) \setminus I_\ell$  of degree  $b$ .

The inclusion (55) implies that

$$a_{\lambda_1} \text{in}_\nu \mathbf{Q}^{\lambda_1} + a_{\lambda_2} \text{in}_\nu \mathbf{Q}^{\lambda_2} + \cdots + a_{\lambda_s} \text{in}_\nu \mathbf{Q}^{\lambda_s} = 0. \quad (56)$$

in  $P_b/P_{b+}$ .

By definition of  $I_\ell$ , there exists an element  $g \in I_\ell$  of the form  $X^\epsilon + \sum_p c_p X^{\epsilon_p}$  and a monomial  $X^\delta$  with  $\epsilon_p > \epsilon$  for all  $p$  and  $\lambda_1 = \epsilon + \delta$ . Then, as  $g \in I_\ell \subset \ker(\phi_\ell)$ , we have  $h - a_{\lambda_1} X^\delta g \in \ker(\phi_\ell)$  and the greatest monomial of  $h - a_{\lambda_1} X^\delta g$  is strictly bigger than  $X^{\lambda_1}$ . This contradicts the choice of  $h$ .  $\square$

**Corollary 1.4.6** *Take an element  $\gamma \in \Phi$ ,  $\gamma < \gamma_\lambda$ . The valuation ideal  $P_\gamma$  is generated by all the generalized monomials of value greater than or equal to  $\gamma$  in  $\{Q \mid (Q, Ex(Q)) \in \Psi(\gamma_\lambda)\}$ . The ideal  $P_{\gamma_\lambda}$  is generated by all the generalized monomials of value greater than or equal to  $\gamma_\lambda$  in  $\{Q \mid (Q, Ex(Q)) \in \Psi(\gamma_\lambda) \cup \Theta(\gamma_\lambda)\}$ .*

Proof : Let  $f \in P_\gamma$  (resp.  $f \in P_{\gamma_\lambda}$ ). By the very definition of the standard form of level  $\ell$  such that  $\gamma_\ell = \gamma$ ,  $f$  can be written as an  $A$ -linear combination of generalized monomials of value greater than or equal to  $\gamma$  in  $\{Q \mid (Q, Ex(Q)) \in \Psi(\gamma_\lambda)\}$  (resp.  $\in \Psi(\gamma_\lambda) \cup \Theta(\gamma_\lambda)$ ). Thus  $P_\gamma$  (resp.  $P_{\gamma_\lambda}$ ) is generated by the generalized monomials of value at least  $\gamma$ , as desired.  $\square$

## 1.5 Approximate roots in a complete local ring

We now generalize the notion of approximate root to a **complete** regular local ring  $A$  of dimension  $n$ , with maximal ideal  $\mathfrak{m}$ , and residue field  $k = \frac{A}{\mathfrak{m}}$ . Let  $\mathbf{u} = (u_1, \dots, u_n)$  be a regular system of parameters and

$$\nu : A \setminus \{0\} \rightarrow \Gamma$$

a valuation, centered in  $\mathfrak{m}$ . Denote by  $\nu_{\mathfrak{m}}$  the  $\mathfrak{m}$ -adic valuation.

We keep the same notation as in §2.

The purpose of this section is to construct, for a general  $\nu$ , a system of approximate roots of  $\nu$ , that is, a well-ordered collection of elements  $\mathbf{Q} = \{Q_i\}_{i \in \Lambda}$  of  $A$  such that for every  $\nu$ -ideal  $I$  in  $A$ , we have

$$I = \left\{ \prod_j Q_j^{\gamma_j} \mid \sum_j \gamma_j \nu(Q_j) \geq \nu(I) \right\} A \quad (57)$$

(in particular, the images  $\text{in}_{\nu} Q_i$  of the  $Q_i$  in  $G$  generate  $G$  as a  $k$ -algebra). Each  $Q_{i+1}$  will be described by an explicit formula (given later in this section) in terms of the  $Q_j$ ,  $j < i$ .

In this general setting, we have to proceed by transfinite induction on the well-ordered semigroup  $\Phi$ . Since we are not assuming that  $\text{rk } \Gamma = 1$  or that  $\Phi$  is archimedean, we have to work with ordinals other than the natural numbers.

*Remark on the use of transfinite induction.* Since the ring  $A$  is noetherian, the group  $\Gamma$  of values of  $\nu$  has finite rank. Therefore all the ordinals  $\ell$  we will encounter in this paper will be of type  $\ell \leq \omega^n$  (cf. [43] and [8]). Thus we will be using a very special form of transfinite induction, which amounts to usual induction, applied finitely many times. We will, however, stick to the language of transfinite induction since we feel that it simplifies the exposition.

Recall the definition of generalized monomial (Definition 1.3.2). We well-order the set of generalized monomials by the lexicographical order on the set of triples  $(\nu(\mathbf{Q}^{\alpha}), \nu_{\mathfrak{m}}(\mathbf{Q}^{\alpha}), \alpha)$ .

The semigroup  $\Phi$  is well ordered. By abuse of notation, we will sometimes write  $\Phi$  for the ordinal given by the order type of  $\Phi$ . Let  $\lambda < \Phi$  be an ordinal and  $\gamma_{\lambda}$  the element of  $\Phi$  corresponding to  $\lambda$ .

We start by choosing a coordinate system adapted to the situation. Fix an isomorphism

$$A \cong k[[u_1, \dots, u_n]]. \quad (58)$$

**Definition 1.5.1** Take  $j \in \{2, \dots, n\}$ . We say that  $u_j$  is  $\nu$ -prepared if there does not exist  $f \in A$  such that  $\text{in}_{\nu} u_j = \text{in}_{\nu} f$  and  $f \in k[[u_1, \dots, u_{j-1}]]$ . The coordinate system  $\mathbf{u} = \{u_1, \dots, u_n\}$  is  $\nu$ -prepared if  $u_j$  is  $\nu$ -prepared for all  $j \in \{2, \dots, n\}$ .

**Proposition 1.5.2** There exists a  $\nu$ -prepared coordinate system.

Proof : We construct a  $\nu$ -prepared coordinate system recursively in  $j$ . Assume that  $u_1, \dots, u_{j-1}$  are  $\nu$ -prepared, but  $u_j$  is not.

We will construct the prepared coordinate  $\tilde{u}_j$  recursively by transfinite induction on  $\Phi$ . More precisely, we will construct a well ordered set  $\{u_{j,i}\}$  of successive approximation to  $\tilde{u}_j$  in the  $\mathfrak{m}$ -adic topology. We will show that this set satisfies the hypothesis of Zorn's lemma and let  $\tilde{u}_j$  be its maximal element.

The details go as follows. Let  $u_{j0} = u_j$ . Suppose that  $u_{ji}$  is constructed and that it is not prepared. Let  $f_{ji}$  be the element  $f$  of  $k[[u_1, \dots, u_{j-1}]]$  appearing in the definition of “not prepared”. Put  $u_{j,i+1} = u_{ji} - f_{ji}$ . Then  $\nu(u_{ji}) = \nu(f_{ji}) < \nu(u_{j,i+1})$ . Next, suppose given a sequence  $u_{ji}, u_{j,i+1}, \dots$  of elements of  $k[[u_1, \dots, u_j]]$  such that  $(u_1, \dots, u_{j-1}, u_{jq})$  is a regular system of parameters of  $k[[u_1, \dots, u_j]]$  for each  $q$  and  $\nu(u_{ji}) < \nu(u_{j,i+1}) < \nu(u_{j,i+2}) < \dots$ . Let  $\beta_q = \nu(u_{jq})$ . Let  $\bar{\beta} = \min\{\beta \in \Phi \mid \beta > \beta_q, \forall q \in \mathbb{N}\}$ . By Chevalley’s lemma, applied to the nested sequence of ideals  $\frac{P_{\beta_q} \cap k[[u_1, \dots, u_{j-1}]]}{P_{\bar{\beta}} \cap k[[u_1, \dots, u_{j-1}]]}$  in the complete local ring  $\frac{k[[u_1, \dots, u_{j-1}]]}{P_{\bar{\beta}} \cap k[[u_1, \dots, u_{j-1}]]}$ , we see that  $\lim_{q \rightarrow \infty} (f_{jq} \bmod P_{\bar{\beta}}) = 0$  in the  $(u_1, \dots, u_{j-1})$ -adic topology.

Hence, modifying each  $f_{jq}$  by an element of  $P_{\bar{\beta}}$  if necessary, we may assume that

$$\lim_{q \rightarrow \infty} f_{jq} = 0.$$

We define  $u_{j,i+\omega}$  to be the formal power series  $u_{ji} - f_{ji} - f_{j,i+1} - \dots$ . By construction,

$$\nu(u_{j,i+\omega}) \geq \bar{\beta}.$$

To complete our construction, we need to consider countable well ordered sets  $\{u_{jt}\}$  of order type greater than  $\omega$ . This presents no problem: by countability, we can always choose a cofinal subsequence in each such set. Then the above construction applies verbatim.  $\square$

We construct, inductively in  $\lambda$ , two well-ordered sets  $\Lambda(\gamma_\lambda)$  and  $\Theta(\gamma_\lambda)$ . At each step we define two additional well-ordered sets  $\mathcal{V}(\gamma_\lambda) \subset \Psi(\gamma_\lambda) \subset \Lambda(\gamma_\lambda)$ . A typical element of each those sets will have the form  $(Q, \text{Ex}(Q))$  where  $Q \in A$  and  $\text{Ex}(Q)$  is an increasing sum of monomials in  $\mathcal{V}(\gamma_\lambda)$  (according to the order we defined on monomials). Given an element  $(Q, \text{Ex}(Q)) \in \Lambda(\gamma_\lambda) \cup \Theta(\gamma_\lambda)$ ,  $Q$  is called an *approximate root* and  $\text{Ex}(Q)$  is called the *expression* of  $Q$ .

For an ordinal  $\ell < \Phi$  and for  $(Q, \text{Ex}(Q)) \in \Lambda(\gamma_\ell) \cup \Theta(\gamma_\ell)$ , let  $\text{In } Q$  denote the smallest monomial of  $\text{Ex}(Q)$ . Let  $\text{In}(\ell) = \{\alpha \in \mathbb{N}^{\mathcal{V}(\gamma_\ell)} \mid \exists Q \in \Lambda(\gamma_\ell) \text{ such that } \mathbf{Q}^\alpha = \text{In } Q\}$ .

**Theorem 1.5.3** *For  $\lambda < \Phi$ , there exist well ordered sets  $\mathcal{V}(\gamma_\lambda) \subset \Psi(\gamma_\lambda) \subset \Lambda(\gamma_\lambda)$  and  $\Theta(\gamma_\lambda)$  having the following properties. Let*

$$\Psi(< \gamma_\lambda) = \Psi(\gamma_{\lambda-1}) \quad \text{if } \lambda \text{ is not a limit ordinal and} \quad (59)$$

$$\Psi(< \gamma_\lambda) = \Psi(\gamma_\lambda) \quad \text{otherwise} \quad (60)$$

and similarly for  $\mathcal{V}(< \gamma_\lambda)$ . Then each set  $\mathcal{V}(\gamma_\lambda), \Psi(\gamma_\lambda), \Lambda(\gamma_\lambda), \Theta(\gamma_\lambda)$  consists of elements of the form  $(Q, \text{Ex}(Q))$ , with  $Q \in A$  and  $\text{Ex}(Q)$  is an increasing, with respect to the monomial order, sum of monomials in  $\mathcal{V}(< \gamma_\lambda)$ , of value  $< \nu(Q)$ , such that

$$\nu(Q) < \gamma_\lambda \text{ whenever } (Q, \text{Ex}(Q)) \in \Lambda(\gamma_\lambda) \quad (61)$$

$$\nu(Q) \geq \gamma_\lambda \text{ whenever } (Q, \text{Ex}(Q)) \in \Theta(\gamma_\lambda) \quad (62)$$

and the sets

$$\{(Q, \text{Ex}(Q)) \in \Theta(\gamma_\lambda) \cup \Lambda(\gamma_\lambda) \mid \nu(Q) = \gamma\}, \quad \gamma \in \Phi \quad (63)$$

and

$$\{(Q, \text{Ex}(Q)) \in \Psi(\gamma_\lambda) \cup \Theta(\gamma_\lambda) \mid Q \notin \mathfrak{m}^n\}, \quad n \in \mathbb{N} \quad (64)$$

are finite. An element  $(Q, \text{Ex}(Q)) \in \Psi(\gamma_\lambda) \cup \Theta(\gamma_\lambda)$  is completely determined by  $\text{In } Q$ ; moreover  $\nu_{\mathfrak{m}}(\text{In } Q) = \nu_{\mathfrak{m}}(Q)$ .



In what follows,  $\Lambda(< \gamma_\lambda)$  will stand for  $\bigcup_{\ell < \lambda} \Lambda(\gamma_\ell)$ .

**Remark 1.5.4** The set  $\bigcap_{\ell \leq \ell' < \lambda} \Psi(\gamma_{\ell'})$  describes the approximate roots which could be called stably essential between  $\gamma_\ell$  and  $\gamma_\lambda$ .

Proof : We proceed by transfinite induction.

First define  $\Psi(\mathbf{1}) = \Lambda(\mathbf{1}) = \emptyset$  and  $\Theta(\mathbf{1}) = \{(u_1, u_1), \dots, (u_n, u_n)\}$  where we assume  $\nu(u_1) \leq \nu(u_2) \leq \dots \leq \nu(u_n)$ .

Let  $\lambda < \Phi$  be an ordinal. Assume that for each  $\ell < \lambda$  we have constructed sets  $\Psi(\gamma_\ell) \subset \Lambda(\gamma_\ell)$  and  $\Theta(\gamma_\ell)$  having the properties required in the theorem.

Let

$$\Lambda(\gamma_\lambda) = \Lambda(< \gamma_\lambda) \text{ if } \lambda \text{ is a limit ordinal} \quad (65)$$

$$\Lambda(\gamma_\lambda) = \Lambda(\gamma_{\lambda-1}) \cup \{(Q, \text{Ex}(Q)) \in \Theta(\gamma_{\lambda-1}) \mid \nu(Q) < \gamma_\lambda\} \text{ otherwise.} \quad (66)$$

**Definition 1.5.5** An element  $(Q, \text{Ex}(Q)) \in \Lambda(\gamma_\lambda)$  is an *inessential predecessor* of a root  $(Q', \text{Ex}(Q')) \in \Lambda(\gamma_\lambda)$  if  $\text{Ex}(Q') = \text{Ex}(Q) + \sum_{\alpha} c_{\alpha} \mathbf{Q}^{\alpha}$ , where each  $c_{\alpha}$  is a unit in  $A$  and  $\mathbf{Q}^{\alpha}$  a monomial in  $\mathcal{V}(\gamma_\lambda)$ .

An element  $(Q, \text{Ex}(Q)) \in \Lambda(\gamma_\lambda)$  is said to be *essential at the level*  $\gamma_\lambda$  if  $Q$  is not an inessential predecessor of an element of  $\Lambda(\gamma_\lambda)$ .

**Remark 1.5.6** Note that, because the system of coordinates is prepared,  $u_1, \dots, u_n$  are always essential.

Let  $\Psi(\gamma_\lambda)$  be the subset of  $\Lambda(\gamma_\lambda)$  consisting of all the essential roots at the level  $\gamma_\lambda$ . Let  $\mathcal{V}(\gamma_\lambda)$  be the subset of  $\Psi(\gamma_\lambda)$  consisting of all  $(Q, \text{Ex}(Q))$  such that  $\text{in}_{\nu}(Q)$  does not belong to the  $k$ -vector space of  $\text{gr}_{\nu}(A)$  generated by the set  $\{\text{in}_{\nu} \mathbf{Q}^{\gamma}\}$  where  $\mathbf{Q}^{\gamma}$  runs over the set of all the generalized monomials on roots preceding  $Q$  in the above ordering.

We extend the total ordering from  $\Lambda(< \gamma_\lambda)$  to  $\Lambda(\gamma_\lambda)$  by postulating that  $\Lambda(< \gamma_\lambda)$  is the initial segment of  $\Lambda(\gamma_\lambda)$ .

If  $\ell$  is not a limit ordinal, let  $E(\ell) = \text{In}(\ell) + \mathbb{N}^{\Lambda(\gamma_\ell)} \subset \mathbb{N}^{\Lambda(\gamma_\ell)}$ . Now, if  $\ell' < \ell''$ ,  $\Lambda(\gamma_{\ell'}) \subset \Lambda(\gamma_{\ell''})$ , this induces an inclusion  $\mathbb{N}^{\Lambda(\gamma_{\ell'})} \subset \mathbb{N}^{\Lambda(\gamma_{\ell''})}$ . So, if  $\ell$  is a limit ordinal, define  $E(\ell) = \bigcup_{\ell' < \ell} E(\ell')$ .

**Notation** Denote by  $\Theta(< \gamma_\lambda)$  the set  $\bigcup_{\ell < \lambda} \Theta(\gamma_\ell) \setminus \Lambda(< \gamma_\lambda)$ .

**Lemma 1.5.7** The set  $\mathbf{Q}(n) = \{(Q, \text{Ex}(Q)) \in \Psi(\gamma_\lambda) \cup \Theta(< \gamma_\lambda) \mid Q \notin \mathfrak{m}^n\}$  is finite for every  $n \in \mathbb{N}$ .

Proof : By the induction hypothesis, for any  $Q \in \mathbf{Q}(n)$ , we have  $\nu_{\mathfrak{m}}(Q) = \nu_{\mathfrak{m}}(\text{In}Q)$ .

Now let  $Q \in \mathbf{Q}(n)$ . Then its expression is of the form  $\text{Ex}(Q) = \mathbf{Q}^{\alpha_0} + \dots$  where, by the very construction,  $\mathbf{Q}^{\alpha_0}$  is either a  $u_r$  or a product, of at least 2 terms,  $\mathbf{Q}^{\alpha_0} = \prod Q_s^{\beta_s}$ .

In the first case, the number of such  $\mathbf{Q}^{\alpha_0}$  is finite, because the number of  $u_k$  is finite.

In the second case,  $\nu_{\mathfrak{m}}(Q_s) < \nu_{\mathfrak{m}}(\mathbf{Q}^{\alpha_0}) \leq \nu_{\mathfrak{m}}(Q) < n$ . So  $\nu_{\mathfrak{m}}(Q_s) < n - 1$  and, by induction on  $n$ , the number of such  $Q_s$  is finite. If  $m = \min\{\nu_{\mathfrak{m}}(Q_s); Q_s \mid \mathbf{Q}^{\alpha_0}\}$ ,  $|\alpha_0| m \leq \nu_{\mathfrak{m}}(\mathbf{Q}^{\alpha_0}) < n - 1$ , so there is a finite number of such  $\alpha_0$  possible which means that the number of such  $\mathbf{Q}^{\alpha_0}$  is finite. By the induction hypothesis,  $Q$  is completely determined by  $\text{In}Q$  whenever  $(Q, \text{Ex}(Q)) \in \Psi(\gamma_\lambda) \cup \Theta(< \gamma_\lambda)$ . Therefore  $\mathbf{Q}(n)$  is finite.  $\square$

**Corollary 1.5.8** *The set of monomials  $\{\mathbf{Q}^\alpha \mid \mathbf{Q}^\alpha \notin \mathfrak{m}^n\}$  in  $\Psi(\gamma_\lambda) \cup \Theta(< \gamma_\lambda)$  is finite for every  $n \in \mathbb{N}$ .*

**Corollary 1.5.9** (1) *Any infinite sequence of generalized monomials in  $\Psi(\gamma_\lambda) \cup \Theta(< \gamma_\lambda)$ , all of whose members are distinct, converges to 0 in the  $\mathfrak{m}$ -adic topology.*

(2) *Any infinite series, all of whose terms are distinct generalized monomials in  $\Psi(\gamma_\lambda) \cup \Theta(< \gamma_\lambda)$  converges in the  $\mathfrak{m}$ -adic topology.*

**Lemma 1.5.10** *The set*

$$\mathbf{Q}^\alpha = \prod Q^{\alpha_Q}, (Q, \text{Ex}(Q)) \in \Psi(\gamma_\lambda) \cup \{(Q, \text{Ex}(Q)) \in \Theta(< \gamma_\lambda) \mid \nu(Q) = \gamma_\lambda\}$$

*is finite.*

Proof : By the Artin-Rees lemma, there exists  $n_0$  such that, for  $n \geq n_0$ ,

$$\mathfrak{m}^n \cap P_{\gamma_\lambda} = \mathfrak{m}^{n-n_0}(\mathfrak{m}^{n_0} \cap P_{\gamma_\lambda}).$$

Take  $n > n_0$ , then

$$\mathfrak{m}^n \cap P_{\gamma_\lambda} \subset \mathfrak{m}P_{\gamma_\lambda} \subset P_{\gamma_\lambda+}. \quad (67)$$

Together with the above corollary, this proves the lemma.  $\square$

Consider now the ordered set  $\{\mathbf{Q}^{\alpha_1}, \dots, \mathbf{Q}^{\alpha_s}\}$  of monomials

$$\mathbf{Q}^\alpha = \prod Q^{\alpha_Q}, (Q, \text{Ex}(Q)) \in \Psi(\gamma_\lambda) \cup \{(Q, \text{Ex}(Q)) \in \Theta(< \gamma_\lambda) \mid \nu(Q) = \gamma_\lambda\}, \alpha \notin E(\lambda)$$

of value  $\gamma_\lambda$ . The fact that this set is **finite** follows from the above Corollaries and the fact that  $\Phi$  is well-ordered.

Let  $i_1 = \max\{i \in \{1, \dots, s\} \mid \text{in}_\nu(\mathbf{Q}^{\alpha_i}) \in \sum_{j=i+1}^s k \text{in}_\nu(\mathbf{Q}^{\alpha_j})\}$  and consider the unique relation  $\text{in}_\nu(\mathbf{Q}^{\alpha_{i_1}}) - \sum_{j=i_1+1}^s c_{1j} \text{in}_\nu(\mathbf{Q}^{\alpha_j}) = 0$ . Let  $P_1 = \mathbf{Q}^{\alpha_{i_1}} - \sum_{j=i_1+1}^s c_{1j} \mathbf{Q}^{\alpha_j}$  where we view  $k$  as a subring of  $A$  via the identification (58).

Let  $i_2 = \max\{i \in \{1, \dots, i_1 - 1\} \mid \text{in}_\nu(\mathbf{Q}^{\alpha_i}) \in \sum_{j=i+1}^s k \text{in}_\nu(\mathbf{Q}^{\alpha_j})\}$  and, as before, consider the unique  $P_2 = \mathbf{Q}^{\alpha_{i_2}} - \sum_{\substack{j=i_2+1 \\ j \neq i_1}}^s c_{2j} \mathbf{Q}^{\alpha_j}$  such that the vector  $(\alpha_j)_{j=i_1+1, \dots, s}$ ,  $c_{2j} \neq 0$ , is minimal in the lexicographical order and define so on uniquely  $P_3, \dots, P_t$ .

Now, if  $\lambda$  has a predecessor, we let

$$\Theta(\gamma_\lambda) = \{(Q, \text{Ex}(Q)) \in \Theta(< \gamma_\lambda) \mid \nu(Q) > \gamma_\lambda\} \cup \{(P_1, \text{Ex}(P_1)), \dots, (P_t, \text{Ex}(P_t))\} \quad (68)$$

where

$$\text{Ex}(P_j) = \mathbf{Q}^{\alpha_{i_j}} - \sum_k c_{jk} \mathbf{Q}^{\alpha_k} \quad (69)$$

if  $\mathbf{Q}^{\alpha_{i_j}}$  is not a preceding root  $Q$  and

$$\text{Ex}(P_j) = \text{Ex}(Q) - \sum_k c_{jk} \mathbf{Q}^{\alpha_k} \quad (70)$$

in the other case.

**Remark 1.5.11** *Note that the remark (1.3.8) remains valid in this context.*

Suppose now  $\lambda$  is a limit ordinal. Let  $Q_0 \in \Lambda(\gamma_{\ell_0})$  for some  $\ell_0 < \lambda$  and  $\mathbf{Q}^\alpha = \text{In}(Q_0)$ . Let  $L(Q_0)$  denote the union of well ordered sets of the form  $\{(Q, \text{Ex}(Q)) \mid (Q, \text{Ex}(Q)) \in \Lambda(\gamma_\ell) \text{ and } \text{In}(Q) = \mathbf{Q}^\alpha\}_{\ell_0 \leq \ell < \lambda}$  such that, for some  $\ell' > \ell$ ,  $\ell' < \lambda$ , there exists  $Q' \in \Lambda(\gamma_{\ell'})$  and

$$\text{Ex}(Q') = \text{Ex}(Q) + \sum_{j \in W} c_j \mathbf{Q}^{\alpha_j} \quad (71)$$

where  $\nu(\mathbf{Q}^{\alpha_j}) \geq \nu(Q)$ . The set  $L(Q_0)$  is an infinite well ordered set.

Let  $n$  be a positive integer. By induction assumption, all the approximate roots  $Q$  appearing in any of the monomials  $\mathbf{Q}^{\alpha_j}$  belong to  $\mathcal{V}(\gamma_\lambda)$  and, by lemma 1.5.7, the number of such roots outside  $\mathfrak{m}^n$  is finite. Thus, all but finitely many  $\mathbf{Q}^{\alpha_j}$  belong to  $\mathfrak{m}^n$ . This proves that  $L(Q_0)$  has a limit in  $A$  with respect to the  $\mathfrak{m}$ -adic topology :  $(\lim_{\rightarrow} Q, \lim_{\rightarrow} \text{Ex}(Q))$ .

Let

$$\Theta(\gamma_\lambda) = \{(Q, \text{Ex}(Q)) \in \Theta(< \gamma_\lambda) \mid \nu(Q) \geq \gamma_\lambda\} \cup \hat{L} \quad (72)$$

where  $\hat{L}$  consists of all couples of the form  $(\lim_{\rightarrow} Q, \lim_{\rightarrow} \text{Ex}(Q))$ .

So finally, the expression of an approximate root has the form

$$\text{Ex}(Q) = \mathbf{Q}^{\alpha_{i_j}} + \sum_k a_k \mathbf{Q}^{\alpha_k} \quad (73)$$

the sum, written in the increasing order of the monomials, being finite or infinite.

We now prove the finiteness of sets (63) and (64). First, note that the set

$$\{(Q, \text{Ex}(Q)) \in \Theta(< \gamma_\lambda) \cup \Lambda(\gamma_\lambda) \mid \nu(Q) = \gamma\}, \quad \gamma \in \Phi \quad (74)$$

is finite by the induction hypothesis and the set

$$\{(Q, \text{Ex}(Q)) \in \Psi(\gamma_\lambda) \cup \Theta(< \gamma_\lambda) \mid Q \notin \mathfrak{m}^n\}, \quad n \in \mathbb{N} \quad (75)$$

is finite by the induction hypothesis and lemma (1.5.10). If  $\lambda$  is not a limit ordinal, the finiteness of (63) and (64) follows from the fact that the set  $\Theta(\gamma_\lambda) \setminus \Theta(< \gamma_\lambda)$  is finite by construction. If  $\lambda$  is a limit ordinal, to prove finiteness of (63) and (64), it remains to prove that the set

$$\{(Q, \text{Ex}(Q)) \in \hat{L} \mid \nu(Q) = \gamma\} \quad (76)$$

is finite. This is proved in exactly the same way as lemma (1.5.10). This completes the proof of the finiteness of (63) and (64).

The property that the monomials appearing in  $\text{Ex}(Q)$  are arranged in increasing order with respect to the  $\nu$ -adic value holds for all the newly constructed approximate roots. Next we show that  $\nu_{\mathfrak{m}}(\text{In}Q) = \nu_{\mathfrak{m}}(Q)$  for all those new approximate roots. Indeed, if  $\lambda$  is not a limit ordinal and  $\text{Ex}(Q)$  is given by formula (69), all the monomials appearing in  $\text{Ex}(Q)$  have the same  $\nu$ -adic value and their  $\nu_{\mathfrak{m}}$ -adic values are increasing because of the order we imposed on monomials which proves that  $\nu_{\mathfrak{m}}(\text{In}Q) = \nu_{\mathfrak{m}}(Q)$ . If  $\text{Ex}(Q')$  is given by formula (70), where  $Q = \mathbf{Q}^{\alpha_0} + \dots$ , we have  $Q' = Q + \sum c_\alpha \mathbf{Q}^\alpha$ , where  $\nu(\mathbf{Q}^\alpha) = \nu(Q)$ . Then  $\nu_{\mathfrak{m}}(Q) \leq \nu_{\mathfrak{m}}(\mathbf{Q}^\alpha)$  for all  $\alpha$ , because of the order on monomials. So that finally,  $\nu_{\mathfrak{m}}(\mathbf{Q}^{\alpha_0}) \leq \nu_{\mathfrak{m}}(Q) \leq \nu_{\mathfrak{m}}(\mathbf{Q}^\alpha)$ , which proves that  $\nu_{\mathfrak{m}}(\text{In}Q) = \nu_{\mathfrak{m}}(Q)$ . The property that  $\nu_{\mathfrak{m}}(\text{In}Q) = \nu_{\mathfrak{m}}(Q)$  is clearly preserved by passing to the limit, so it also holds in the case when  $\lambda$  is a limit ordinal.

**Remark 1.5.12** *We just showed that there is a one to one correspondence between the approximate roots  $Q \in \Psi(\gamma_\ell)$  and the set of monomials which are the first term of the expression  $\text{Ex}(Q)$  of such an approximate root  $Q$ . Let us denote by  $\mathbf{M}(\ell)$  the set of those monomials.*

The rest of theorem (1.5.3) holds by construction.

## 1.6 Standard form in the case of complete regular local rings

Let  $\Psi(\gamma_\Phi) = \bigcup_{\ell < \Phi} \bigcap_{\ell \leq \ell' < \Phi} \Psi(\gamma_{\ell'})$  and let  $\mathcal{V}(\gamma_\Phi)$  be the set of approximate roots, essential at the level  $\gamma_\Phi$ .

In this section, we fix an ordinal  $\lambda \leq \Phi$ .

**Definition 1.6.1** *A monomial in  $\Psi(\gamma_\lambda) \cup \Theta(\gamma_\lambda)$  is called standard with respect to  $\lambda$  if all the approximate roots appearing in it belong to  $\mathcal{V}(\gamma_\lambda)$  and it is not divisible by any  $InQ$  where  $Q$  is an approximate root in  $(\Psi(\gamma_\lambda) \cup \Theta(\gamma_\lambda)) \setminus \{(u_1, u_1), \dots, (u_n, u_n)\}$ .*

Take an ordinal  $\ell \leq \lambda$ .

**Definition 1.6.2** *Let  $f \in A$ . An expansion of  $f$  of the form  $f = \sum c_\alpha \mathbf{Q}^\alpha$  where the  $\mathbf{Q}^\alpha$  are monomials in  $\Psi(\gamma_\lambda) \cup \Theta(\gamma_\lambda)$ , written in increasing order, is a standard form of level  $\gamma_\ell$  if  $\forall \gamma' < \gamma_\ell$  and for all  $\alpha$  such that  $\nu(\mathbf{Q}^\alpha) = \gamma'$ ,  $\mathbf{Q}^\alpha$  is a standard monomial.*

We now construct by induction on  $\ell$  a standard form of  $f$  of level  $\gamma_\ell$ . We will write this standard form as

$$f = f_\ell + \sum c_\alpha \mathbf{Q}^\alpha$$

where, for all  $\alpha$ ,  $\mathbf{Q}^\alpha$  is a standard monomial in  $\Psi(\gamma_\lambda) \cup \Theta(\gamma_\lambda)$ ,  $\nu(\mathbf{Q}^\alpha) \geq \gamma_\ell$  and  $f_\ell$  is a sum of generalized monomials in  $\mathcal{V}(\gamma_\lambda)$  of value strictly less than  $\gamma_\ell$ .

To start the induction, let  $f_0 = 0$ . The *standard form of  $f$  of level 0* will be its expansion,  $f = f_0 + \sum c_\alpha \mathbf{u}^\alpha$ , written in increasing order according to the monomial order defined above, as a formal power serie in the  $u_i$ .

Let  $\ell < \lambda$  be an ordinal. Let us define  $f_{\ell+1}$  and the standard form of  $f$  of level  $\gamma_{\ell+1}$  as follows. Assume, inductively, that a standard form of level  $\gamma_\ell$  is already defined :  $f = f_\ell + \sum c_\alpha \mathbf{Q}^\alpha$  with  $\nu(\mathbf{Q}^\alpha) \geq \gamma_\ell$ , for all  $\alpha$ , and the value of any monomial of  $f_\ell < \gamma_\ell$ .

Take the homogeneous part of  $\sum c_\alpha \mathbf{Q}^\alpha$  of value  $\gamma_\ell$ , the monomials being written in increasing order. Assume that not all the  $\mathbf{Q}^\alpha$  are standard with respect to  $\lambda$ , and take the smallest such  $\mathbf{Q}^\alpha$ . Since  $\mathbf{Q}^\alpha$  is not standard, one of the two following conditions holds :

1. There exists an approximate root  $Q \in (\Psi(\gamma_\lambda) \cup \Theta(\gamma_\lambda)) \setminus \{(u_1, u_1), \dots, (u_n, u_n)\}$  such that  $In(Q)$  divides  $\mathbf{Q}^\alpha$ . Write  $Q = In(Q) + \sum c_\beta \mathbf{Q}^\beta$  and replace  $In(Q)$  by  $Q - \sum c_\beta \mathbf{Q}^\beta$  in  $\mathbf{Q}^\alpha$ .
2. An approximate root  $Q \in \Psi(\gamma_\lambda) \setminus \mathcal{V}(\gamma_\lambda)$  divides  $\mathbf{Q}^\alpha$ . Since  $Q \notin \mathcal{V}(\gamma_\lambda)$ , there exists

$$Q' \in \Psi(\gamma_\lambda) \cup \Theta(\gamma_\lambda)$$

of the form  $Q' = Q + \sum_\beta d_\beta \mathbf{Q}^\beta$ , where the  $\mathbf{Q}^\beta$  are monomials in  $\mathcal{V}(\gamma_\lambda)$  of value greater than or equal to  $\gamma_\ell$ . Replace  $Q$  by  $Q' - \sum_\beta d_\beta \mathbf{Q}^\beta$ .

In both cases, those changes introduce new monomials, with increasing  $\nu_m$  value, but either they are of value strictly greater than  $\gamma_\ell$  or they are of value exactly  $\gamma_\ell$  but greater than  $\mathbf{Q}^\alpha$  in the monomial ordering. We repeat this procedure as many times as we can. After a finite number of steps, no more changes are available at level  $\gamma_{\ell+1}$ . Then, let  $f_{\ell+1} = f_\ell + \sum d_\rho \mathbf{Q}^\rho$  with  $\nu(\mathbf{Q}^\rho) = \gamma_\ell$ , so that  $f = f_{\ell+1} + \sum c_\alpha \mathbf{Q}^\alpha$  where  $\nu(\mathbf{Q}^\alpha) > \gamma_\ell$ .

Suppose now that  $\mu$  is a limit ordinal. For each  $\ell < \mu$ , write  $f = f_\ell + \delta_\ell$  where  $f_\ell$  is a sum of standard monomials, with respect to  $\lambda$ , of value  $< \gamma_\ell$  and  $\delta_\ell$  is a sum of monomials in  $\Psi(\gamma_\lambda) \cup \Theta(\gamma_\lambda)$ , of value greater than or equal to  $\gamma_\ell$ . We assume inductively that, for each  $\ell < \mu$  and for each generalized monomial  $\mathbf{Q}^\tau$  in  $\Psi(\gamma_\lambda) \cup \Theta(\gamma_\lambda)$ , there exist  $c_\tau, b_\tau \in k$  and

an ordinal  $\ell_0 < \ell$  such that, for all  $\ell'$ ,  $\ell_0 < \ell' < \ell$ , the monomial  $\mathbf{Q}^\tau$  appears in  $f_{\ell'}$  with coefficient  $c_\tau$  and in  $\delta_{\ell'}$  with coefficient  $b_\tau$ . Moreover, assume that  $f_\ell = \lim_{\ell' < \ell} f_{\ell'} = \sum_\tau c_\tau \mathbf{Q}^\tau$  and  $\delta_\ell = \lim_{\ell' < \ell} \delta_{\ell'} = \sum_\tau b_\tau \mathbf{Q}^\tau$ .

**Lemma 1.6.3** *Consider a generalized monomial  $\mathbf{Q}^\tau$  in  $\Psi(\gamma_\lambda) \cup \Theta(\gamma_\lambda)$ . There exist  $c_\tau, b_\tau \in k$  and an ordinal  $\ell_0 < \mu$  such that, for all  $\ell$ ,  $\ell_0 < \ell < \mu$ , the monomial  $\mathbf{Q}^\tau$  appears in  $f_\ell$  with coefficient  $c_\tau$  and in  $\delta_\ell$  with coefficient  $b_\tau$ .*

**Corollary 1.6.4** *The limits  $\lim_{\ell < \mu} f_\ell$  and  $\lim_{\ell < \mu} \delta_\ell$  exist in the  $\mathfrak{m}$ -adic topology.*

This is an immediate consequence of the Lemma and Corollary 1.5.9.

For each  $\mathbf{Q}^\tau$  as above, let  $c_\tau, b_\tau$  be as in Lemma 1.6.3. Let  $f_\mu = \lim_{\ell < \mu} f_\ell = \sum_\tau c_\tau \mathbf{Q}^\tau$  and  $\delta_\mu = \lim_{\ell < \mu} \delta_\ell = \sum_\tau b_\tau \mathbf{Q}^\tau$ . We define the standard form of  $f$  of level  $\gamma_\mu$  as  $f = f_\mu + \delta_\mu$ .

Proof of Lemma 1.6.3 : The existence of  $c_\tau$  in the lemma follows immediately from the construction and the induction hypothesis.

If  $\nu(\mathbf{Q}^\tau) < \gamma_\mu$ , put  $b_\tau = 0$ . For  $\ell < \mu$ , let  $b_\tau(\ell)$  denote the coefficient of  $\mathbf{Q}^\tau$  in  $\delta_\ell$ . Take an ordinal  $\ell < \mu$ . Suppose

$$b_\tau(\ell) \neq b_\tau(\ell + 1). \quad (77)$$

This means that in the above construction of  $f_{\ell+1} + \delta_{\ell+1}$  from  $f_\ell + \delta_\ell$ ,  $\mathbf{Q}^\tau$  appears in one of the expressions  $\frac{\mathbf{Q}^\alpha}{InQ}Q, \frac{\mathbf{Q}^\alpha}{InQ} \sum_\beta d_\beta \mathbf{Q}^\beta$  (case 1 of the construction) or  $\frac{\mathbf{Q}^\alpha}{Q}Q', \frac{\mathbf{Q}^\alpha}{Q} \sum_\beta d_\beta \mathbf{Q}^\beta$  (case 2 of the construction). Then

$$\nu_{\mathfrak{m}}(\mathbf{Q}^\alpha) \leq \nu_{\mathfrak{m}}(\mathbf{Q}^\tau). \quad (78)$$

Suppose that there were infinitely many  $\ell$  for which (77) holds. This would mean that there are infinitely many monomials  $\mathbf{Q}^\alpha$  (all distinct because  $\nu(\mathbf{Q}^\alpha) = \gamma_\ell$ ), satisfying (78). This contradicts Lemma 1.5.7; hence there are finitely many such  $\ell$ . Together with the induction hypothesis, this proves that  $b_\tau(\ell)$  stabilizes for  $\ell$  sufficiently large. This completes the proof of the lemma.  $\square$

This completes the construction of standard form of level  $\gamma_\ell$  for  $\ell \leq \gamma_\lambda$ .

**Proposition 1.6.5** *Let*

$$f = f_\ell + \sum c_\alpha \mathbf{Q}^\alpha$$

*be a standard form of  $f$  of level  $\gamma_\ell$  and  $\gamma < \gamma_\ell$  an element of  $\Phi$ . Then  $\sum_{\nu(\mathbf{Q}^\beta) = \gamma} c_\beta \mathbf{Q}^\beta \notin P_{\gamma+}$ .*

The proof is entirely the same as the proof of the analogous Proposition 1.4.3.

For each  $\ell$ , the part  $f_\ell$  of a standard form of  $f$  of level  $\gamma_\ell$  is uniquely determined. This is a straightforward consequence of the proposition.

By Proposition 1.6.5, if  $\gamma_\ell > \nu(f)$  then  $\nu(f)$  equals the smallest value of a monomial appearing the standard form of  $f$  of level  $\gamma_\ell$ .

**Theorem 1.6.6** (1) Take  $\gamma \in \Phi$ ,  $\gamma < \gamma_\lambda$ . Then  $\frac{P_\gamma}{P_{\gamma+}}$  is generated as a  $k$ -vector space by  $\{in_\nu \mathbf{Q}^\beta\}$  where  $\mathbf{Q}^\beta$  runs over the set of all standard monomials with respect to  $\lambda$ , satisfying  $\nu(\mathbf{Q}^\beta) = \gamma$ .  
(2) The part of the graded  $k$ -algebra  $gr_\nu(A)$  of degree strictly less than  $\gamma_\lambda$  is generated by the initial forms of the approximate roots of  $\mathcal{V}(\gamma_\lambda)$ .

The same proof as that of Theorem 1.4.4 works here.

Now, for each ordinal  $\ell$ , let  $X = X_{\mathcal{V}(\gamma_\ell)}$  be a set of independent variables, indexed by  $\mathcal{V}(\gamma_\ell)$  and consider the graded  $k$ -algebra  $k[X_{\mathcal{V}(\gamma_\ell)}]$ , where we define  $\deg X_j = \nu(Q_j)$ . Let  $P$  denote the homogeneous monomial ideal of  $k[X_{\mathcal{V}(\gamma_\ell)}]$  generated by all the monomials in  $X_{\mathcal{V}(\gamma_\ell)}$  of degree greater than or equal to  $\gamma_{\ell+1}$ . We have the natural map

$$\phi_\ell : \begin{array}{ccc} \frac{k[X_{\mathcal{V}(\gamma_\ell)}]}{P} & \rightarrow & \frac{gr_\nu A}{P_{\gamma_{\ell+1}}} \\ X_j & \mapsto & in_\nu Q_j \end{array} .$$

Now, for  $\ell = 0$ , let  $I_0 = (0)$ . For  $\ell > 0$ , let  $I_\ell$  denote the ideal of  $\frac{k[X_{\mathcal{V}(\gamma_\ell)}]}{P}$  generated by  $I_{<\ell}$  and all the homogeneous polynomials of the form

$$X^{\alpha_0} + \lambda_1 X^{\alpha_1} + \lambda_2 X^{\alpha_2} + \cdots + \lambda_{b_0} X^{\alpha_{b_0}} \quad (79)$$

where  $\mathbf{Q}^{\alpha_0} + \lambda_1 \mathbf{Q}^{\alpha_1} + \lambda_2 \mathbf{Q}^{\alpha_2} + \cdots + \lambda_{b_0} \mathbf{Q}^{\alpha_{b_0}}$  is the homogeneous part of least degree of  $Ex(Q)$ ,  $Q \in \mathcal{V}(\gamma_\ell) \cup \Theta(\gamma_\ell)$ .

Once again the proves of Corollary 1.4.5 and Corollary 1.4.6 give the analogous corollaries.

**Corollary 1.6.7** We have  $Ker \phi_\ell = I_\ell$ .

**Corollary 1.6.8** Take an element  $\gamma \in \Phi$ ,  $\gamma < \gamma_\lambda$ . The valuation ideal  $P_\gamma$  is generated by all the generalized monomials of value  $\geq \gamma$  in  $\{Q \mid (Q, Ex(Q)) \in \Psi(\gamma_\lambda)\}$ . The ideal  $P_{\gamma_\lambda}$  is generated by all the generalized monomials of value  $\geq \gamma_\lambda$  in  $\{Q \mid (Q, Ex(Q)) \in \Psi(\gamma_\lambda) \cup \Theta(\gamma_\lambda)\}$ .

## Part 2. Separating ideal and convex sets

### 2.1 A description of the separating ideal.

Let  $A$  be a noetherian ring and  $\alpha$  and  $\beta$  points in  $Sper A$ . The purpose of this section is twofolds. First we prove a general result on the behaviour of  $\langle \alpha, \beta \rangle$  under localization. Secondly, we restrict attention to the case when  $A$  is regular and is either complete or  $\langle \alpha, \beta \rangle$  is primary to a maximal ideal of  $A$ . In this case, we describe generators of the separating ideal  $\langle \alpha, \beta \rangle$  as generalized monomials in those approximate roots  $Q_j$  which are common to  $\nu_\alpha$  and  $\nu_\beta$ .

**Proposition 2.1.1** Let  $A$  be a ring. Consider points  $\alpha, \beta \in Sper A$  whose respective supports are  $\mathfrak{p}_\alpha, \mathfrak{p}_\beta$  and let  $\epsilon$  be a common specialization of  $\alpha$  and  $\beta$  with support  $\mathfrak{p}$ .

Then we have  $\langle \alpha, \beta \rangle A_{\mathfrak{p}} = \langle \alpha A_{\mathfrak{p}}, \beta A_{\mathfrak{p}} \rangle$ .

Next, let  $\epsilon \in Sper A$  be the least common specialization of  $\alpha$  and  $\beta$ . Let  $\mathfrak{p} = \mathfrak{p}_\epsilon$ . Then

$$\langle \alpha, \beta \rangle \subset \langle \alpha, \beta \rangle A_{\mathfrak{p}} \cap A. \quad (80)$$

with equality if  $\langle \alpha, \beta \rangle$  is  $\mathfrak{p}$ -primary.

If  $\epsilon$  is the unique common specialization of  $\alpha$  and  $\beta$  (in particular, whenever  $\mathfrak{p} = \sqrt{\langle \alpha, \beta \rangle}$  and  $\mathfrak{p}$  is maximal), then we have equality in (80).

Proof : Let  $f$  be a generator of  $\langle \alpha, \beta \rangle$  such that  $f$  changes sign between  $\alpha$  and  $\beta$ . Say,  $f(\alpha) \geq 0$  and  $f(\beta) \leq 0$ . As the orders on  $A/\mathfrak{p}_\alpha$  and  $A_{\mathfrak{p}}/\mathfrak{p}_\alpha A_{\mathfrak{p}}$  are the same (the quotient field is the same) — and similarly for  $\mathfrak{p}_\beta$  —  $f$  changes sign between  $\alpha A_{\mathfrak{p}}$  and  $\beta A_{\mathfrak{p}}$ . Thus  $f \in \langle \alpha A_{\mathfrak{p}}, \beta A_{\mathfrak{p}} \rangle$ .

Conversely, a generator of  $\langle \alpha A_{\mathfrak{p}}, \beta A_{\mathfrak{p}} \rangle$  is of the form  $g/s$ ,  $s \notin \mathfrak{p}$ , such that  $\frac{g}{s}(\alpha A_{\mathfrak{p}}) \geq 0$  and  $\frac{g}{s}(\beta A_{\mathfrak{p}}) \leq 0$ , for instance. But, as  $\mathfrak{p}$  is a specialisation of  $\alpha$  and  $\beta$  and  $s \notin \mathfrak{p}$ ,  $s$  has the same sign on  $\alpha$  and  $\beta$  (and is non-zero at both points), so  $g$  keeps different signs on  $\alpha$  and  $\beta$  which means that  $g \in \langle \alpha, \beta \rangle$ , and, consequently,  $\frac{g}{s} \in \langle \alpha, \beta \rangle A_{\mathfrak{p}}$ .

The last statement of the Proposition follows immediately from the fact that  $\mathfrak{p}$  is the common center of the valuations  $\nu_\alpha$  and  $\nu_\beta$ .  $\square$

Let  $(A, \mathfrak{m}, k)$  be a regular local ring and  $\alpha$  and  $\beta$  two points of  $\text{Sper}(A)$  having a common specialization  $\epsilon$  whose center is the maximal ideal  $\mathfrak{m}$  of  $A$ . Then  $\nu_\alpha$  and  $\nu_\beta$  are both centered at  $\mathfrak{m}$ .

Let us denote by  $P_{\gamma_{\alpha s}}$ ,  $s = 1, \dots, m$ , (resp.  $P_{\gamma_{\beta s}}$ ,  $s = 1, \dots, m$ ) the well ordered set of  $\nu_\alpha$ -ideals (resp.  $\nu_\beta$ -ideals) containing the separating ideal  $\langle \alpha, \beta \rangle$  and denote by  $\gamma_{\alpha s}$  (resp.  $\gamma_{\beta s}$ ) the  $\nu_\alpha$  value of  $P_{\gamma_{\alpha s}}$  (resp. the  $\nu_\beta$  value of  $P_{\gamma_{\beta s}}$ ). Of course we have  $P_{\gamma_{\alpha s}} = P_{\gamma_{\beta s}}$  for all  $s = 1, \dots, m$ .

Let us consider the approximate roots of the valuation  $\nu_\alpha$  and the corresponding sequences of vectors  $\mathbf{m}_i = (m_{i1}, m_{i2}, \dots, m_{it_{i\alpha}})$ ,  $m_{ij} \in P_{\gamma_{\alpha i}}/P_{\gamma_{\alpha, i+1}}$  which are the initial forms of the monomials  $\mathbf{Q}^{\alpha_{ij}}$  of value  $\gamma_{\alpha i}$ ,  $i = 1, \dots, m$  (see section 1.3 and 1.5). We do the same with  $\nu_\beta$  and write  $\mathbf{n}_1, \mathbf{n}_2, \dots, \mathbf{n}_m$  the corresponding sequences of initial forms. Let  $Q_j(\alpha)$  denote the  $j$ -th approximate root for  $\nu_\alpha$  (in the case when  $A$  is complete  $j$  is an ordinal rather than a natural number); we will denote the monomials in these approximate roots by  $\mathbf{Q}(\alpha)^\gamma$ ; similarly for  $Q_j(\beta)$  and  $\mathbf{Q}(\beta)^\gamma$ .

Now, let  $s_{\alpha h}$  denote the number of monomials in the  $Q_j(\alpha)$ 's of value exactly  $\gamma_{\alpha h}$ ; similarly for  $s_{\beta h}$ .

For a given  $\ell$ , consider the following three conditions  $(1)_\ell$ ,  $(2)_\ell$ ,  $(3)_\ell$  :

$$(1)_\ell \quad s_{\alpha i} = s_{\beta i}, \quad 1 \leq i \leq \ell$$

$(2)_\ell$   $\mathbf{Q}(\alpha)^{\alpha_{ij}} = \mathbf{Q}(\beta)^{\alpha_{ij}}$  for  $1 \leq i \leq \ell$  and  $1 \leq j \leq s_{\alpha i}$  (the set of all generalized monomials of value  $\gamma_{\alpha i}$  and the corresponding of value  $\gamma_{\beta i}$ )

$$(3)_\ell \quad \text{For any } i \leq \ell \text{ and } \bar{\lambda}_1, \dots, \bar{\lambda}_{s_{\alpha i}} \in k, \text{ the sign on } \alpha \text{ of the linear combination } \sum_{j=1}^{s_{\alpha i}} \bar{\lambda}_j m_{ij}$$

is the same as the sign on  $\beta$  of  $\sum_{j=1}^{s_{\alpha i}} \bar{\lambda}_j n_{ij}$  (here we adopt the convention that the sign can be strictly positive, strictly negative or zero) where  $m_{ij}, n_{ij}$  are the initial forms of the monomials  $\mathbf{Q}(\alpha)^{\alpha_{ij}}, \mathbf{Q}(\beta)^{\alpha_{ij}}$  in the graded rings  $\text{gr}_{\nu_\alpha}(A)$ ,  $\text{gr}_{\nu_\beta}(A)$ . Note that if conditions  $(1)_\ell$ – $(3)_\ell$  hold then the set of  $k$ -linear relations among the  $m_{ij}$ ,  $i \leq \ell$ , is the same as the set of  $k$ -linear relations among the  $n_{ij}$ .

Let  $i$  be the smallest ordinal such that at least one of the conditions  $(1)_i$ – $(3)_i$  does not hold.

**Proposition 2.1.2** *We have  $\nu_\alpha(\langle \alpha, \beta \rangle) = \gamma_{\alpha i}$  and  $\nu_\beta(\langle \alpha, \beta \rangle) = \gamma_{\beta i}$ .*

Proof : Let  $M_\alpha$  be the set of all the generalized monomials in  $\mathbf{Q}$ , of value  $\gamma_{\alpha i}$  with respect to  $\nu_\alpha$ . Let  $M_\beta$  be the same kind of set with respect to  $\nu_\beta$ . By definitions, we have  $M_\alpha \neq \emptyset$  and  $M_\beta \neq \emptyset$ . We have the following 2 possibilities :



First, suppose  $M_\alpha \neq M_\beta$ . Say,  $M_\alpha \not\subset M_\beta$ . Take generalized monomials  $\mathbf{Q}^\gamma \in M_\alpha \setminus M_\beta$ , and  $\mathbf{Q}^\delta \in M_\beta$ . Then  $\nu_\alpha(\mathbf{Q}^\gamma) \leq \nu_\alpha(\mathbf{Q}^\delta)$ , but  $\nu_\beta(\mathbf{Q}^\gamma) > \nu_\beta(\mathbf{Q}^\delta)$ .

Then there exists a linear combination, with coefficient in  $(A \setminus \mathfrak{m})$ , of  $\mathbf{Q}^\gamma$  and  $\mathbf{Q}^\delta$ , of value  $\gamma_{\alpha i}$  with respect to  $\nu_\alpha$ , which changes sign between  $\alpha$  and  $\beta$ . This shows that  $\nu_\alpha(\langle \alpha, \beta \rangle) \leq \gamma_{\alpha i}$  in this case.

The second case is  $M_\alpha = M_\beta$  and there exist  $\bar{\lambda}_1, \dots, \bar{\lambda}_{s_{\alpha i}}$  such that the sign on  $\alpha$  of  $\sum_{j=1}^{s_{\alpha i}} \bar{\lambda}_j m_{ij}$  differs from the sign on  $\beta$  of  $\sum_{j=1}^{s_{\alpha i}} \bar{\lambda}_j n_{ij}$  (by assumption, we are in the case  $s_{\alpha i} = s_{\beta i}$ ). By a small perturbation of the  $\bar{\lambda}_j$  (for instance, by adding or subtracting a “small” element of  $k$  to  $\bar{\lambda}_1$ ), we can ensure both that  $\sum_{j=1}^{s_{\alpha i}} \bar{\lambda}_j m_{ij} \neq 0$  in  $gr_{\nu_\alpha} A$  and  $\sum_{j=1}^{s_{\alpha i}} \bar{\lambda}_j n_{ij} \neq 0$  in  $gr_{\nu_\beta} A$ . But this gives an  $f = \sum_{j=1}^{s_{\alpha i}} \lambda_j \mathbf{Q}^{\alpha_{ij}} \in A$  which changes signs between  $\alpha$  and  $\beta$ . We have  $\nu_\alpha(f) = \gamma_{\alpha i}$  (and  $\nu_\beta(f) = \gamma_{\beta i}$ ), so  $\nu_\alpha(\langle \alpha, \beta \rangle) \leq \gamma_{\alpha i}$  also in this case.

Now take an  $f \in A$  with  $\nu_\alpha(f) < \gamma_{\alpha i}$ . Then  $f \in P_{\gamma_{\alpha s}}$ ,

$$\gamma_{\alpha s} < \gamma_{\alpha i}, \quad (81)$$

so  $\text{in}_{\nu_\alpha}(f) \in P_{\gamma_{\alpha s}}/P_{\gamma_{\alpha s+}}$ . By theorem 1.6.6,  $\text{in}_{\nu_\alpha}(f)$  is a  $k$ -linear combination of  $m_{s1}, \dots, m_{st_{s\alpha}}$ . By (81) and the definition of  $i$ , this linear combination has the same sign for  $\alpha$  and for  $\beta$  (in other words,  $P_{\gamma_{\alpha s}}/P_{\gamma_{\alpha s+}} = P_{\gamma_{\beta s}}/P_{\gamma_{\beta s+}}$  with same order induced by  $\alpha$  and by  $\beta$ ). This means that  $\text{in}_{\nu_\alpha}(f)$  has the same sign on  $\alpha$  and  $\beta$ , so  $\nu_\alpha(\langle \alpha, \beta \rangle) \geq \gamma_{\alpha i}$ . This completes the proof.

**Corollary 2.1.3** *Let  $\alpha, \beta \in \text{Sper}(A)$ , both centered in the maximal ideal. Let  $i$  be as above. Denote by  $\gamma = \gamma_{\alpha i}$  the  $\nu_\alpha$ -value of  $\langle \alpha, \beta \rangle$ . Let  $Q_1, \dots, Q_{q(i)}$  be the common approximate roots of the valuations  $\nu_\alpha$  and  $\nu_\beta$ . Then  $\langle \alpha, \beta \rangle$  is generated by the generalized monomials in  $Q_1, \dots, Q_{q(i)}$  of  $\nu_\alpha$ -value  $\geq \gamma$  (and the same with  $\nu_\beta$  instead of  $\nu_\alpha$ ).*

Proof : As  $\langle \alpha, \beta \rangle$  is a  $\nu_\alpha$ -ideal (and a  $\nu_\beta$ -ideal), this is a consequence of Corollary 1.6.8.

**Definition 2.1.4** *The approximate roots  $Q_1, \dots, Q_{q(i)}$  (which are common to  $\nu_\alpha$  and  $\nu_\beta$ ) will be called relevant for the pair  $(\alpha, \beta)$ .*

**Definition 2.1.5** *For a graded algebra  $G$ , we define*

$$G^* = \left\{ \frac{f}{g} \mid f, g \in G, g \neq 0 \text{ and homogeneous} \right\} / \sim.$$

where  $\frac{f}{g} \sim \frac{f'}{g'}$  whenever  $fg' = f'g$ .

**The Alvis–Johnston–Madden example.** Let us consider  $\alpha$  and  $\beta$  in  $\text{Sper}(R[x, y, z])$  given by curvettes

$$x(t) = t^6, \quad (82)$$

$$y(t) = t^{10} + ut^{11}, \quad (83)$$

$$z(t) = t^{14} + t^{15} \quad (84)$$

where  $u$  takes 2 distinct values  $u_\alpha > 2$  and  $u_\beta > 2$ . Applying the above procedure, we show that  $\nu_\alpha(\langle \alpha, \beta \rangle) = 31$ .

Indeed, we have  $Q_1 = x, Q_2 = y, Q_3 = z$  for  $\alpha$  and  $\beta$ . The first level approximate roots are

$$Q_4 = y^2 - xz = (2u - 1)t^{21} + u^2t^{22}, \quad (85)$$

$$Q_5 = yz - x^4 = (u + 1)t^{25} + ut^{26}, \quad (86)$$

$$Q_6 = z^2 - x^3y = (2 - u)t^{29} + t^{30}, \quad (87)$$

$$Q_7 = y^3 - x^5 = 3ut^{31} + \dots \quad (88)$$

for both  $\alpha$  and  $\beta$ . Let  $T$  denote the natural image of  $t$  in  $(gr_{\nu_\alpha} R[x, y, z])^*$ , so that

$$(gr_{\nu_\alpha} R[x, y, z])^* \cong (R[T])^*.$$

Then  $\text{in}_{\nu_\alpha}(yQ_4) = (2u_\alpha - 1)T^{31}$  and  $\text{in}_{\nu_\alpha}(xQ_5) = (u_\alpha + 1)T^{31}$ , and similarly for  $\beta$ . Since  $u_\alpha \neq u_\beta$ , the matrix

$$\begin{pmatrix} (2u_\alpha - 1) & (u_\alpha + 1) \\ (2u_\beta - 1) & (u_\beta - 1) \end{pmatrix}$$

is non-singular, so there exists an  $R$ -linear combination of  $\text{in}_{\nu_\alpha}(yQ_4)$  and  $\text{in}_{\nu_\alpha}(xQ_5)$  which is strictly positive on  $\alpha$  and strictly negative on  $\beta$ . According to Proposition 2.1.2,

$$\nu_\alpha(\langle \alpha, \beta \rangle) \leq 31.$$

One can check that 31 is the lowest value for which either there is a linear combination of generalized monomials with this property or the set of monomials of that value for  $\alpha$  does not equal the corresponding set for  $\beta$ , so that in fact  $\nu_\alpha(\langle \alpha, \beta \rangle) = 31$ .

A would-be second level approximate root

$$Q_8 = yQ_4 + \frac{2u - 1}{u + 1}Q_5 \quad (89)$$

is irrelevant because the common value of the generalized monomials appearing on the right hand side of (89) is 31.

## 2.2 Some sets which are conjecturally connected

Let  $(A, \mathfrak{m}, k)$  be a regular local ring. Take  $\alpha, \beta \in \text{Sper}A$ , both centered at  $\mathfrak{m}$ , and elements  $f_1, \dots, f_r \in A \setminus \langle \alpha, \beta \rangle$ . The connected conjecture ([0.1.11]) asserts that there exists a connected set  $C$ , containing  $\alpha, \beta$ , such that  $C$  is disjoint from the zero set of  $f_1 \cdots f_r$ .

Assume that either  $A$  is complete or  $\sqrt{\langle \alpha, \beta \rangle} = \mathfrak{m}$ .

In this section, we describe a set  $C$ , which contains  $\alpha, \beta$ , disjoint from the set  $f_1 \cdots f_r = 0$ , and which we conjecture to be connected. Under the above assumptions, this reduces the connectedness conjecture for  $\alpha$  and  $\beta$  to proving the connectedness of  $C$ .

Let  $\mathbf{Q}_\Lambda = \{Q_\lambda, \lambda \in \Lambda\}$  be the approximate roots common to  $\alpha$  and  $\beta$ . Let  $\mathbf{Q}^{\gamma_1}, \mathbf{Q}^{\gamma_2}, \dots$  be the list of monomials in  $\mathbf{Q}_\Lambda$ , arranged in the increasing order of the  $\nu_\alpha$  values. There exists an ordinal  $s$  such that  $\langle \alpha, \beta \rangle$  is generated by the set  $\{\mathbf{Q}^{\gamma_j}; j \leq s, \mathbf{Q}^{\gamma_j} \in \langle \alpha, \beta \rangle\}$ . Let  $t$  be the unique ordinal such that  $\mathbf{Q}^{\gamma_a} \notin \langle \alpha, \beta \rangle$  for  $a < t$  and  $\mathbf{Q}^{\gamma_t}, \mathbf{Q}^{\gamma_{t+1}}, \dots, \mathbf{Q}^{\gamma_s} \in \langle \alpha, \beta \rangle$ .

Next, we study the standard form of  $f_i$  of level  $\nu_\alpha(\langle \alpha, \beta \rangle)$ . In the case when  $A$  is complete, this standard form may contain infinitely many generalized monomials  $\mathbf{Q}^\gamma$ . Since  $A$  is noetherian, we can choose a finite subset  $\mathbf{Q}^{\epsilon_{j_i}}, 1 \leq j \leq n_i$ , of these monomials such that all of the others lie in the ideal  $(\mathbf{Q}^{\epsilon_{j_i}}, 1 \leq j \leq n_i)A$ . Let  $f_i$ , for  $i \in \{1, \dots, r\}$ ,

$$f_i = \sum_{j=1}^{m_i} b_{j_i} \mathbf{Q}^{\theta_{j_i}} + \sum_{j=1}^{n_i} c_{j_i} \mathbf{Q}^{\epsilon_{j_i}} \quad (90)$$

be the standard expansion of  $f_i$  of level  $\nu_\alpha(\langle \alpha, \beta \rangle)$  where  $\nu_\alpha(\mathbf{Q}^{\theta_{ji}}) = \nu_\alpha(f_i) < \nu_\alpha(\mathbf{Q}^{\epsilon_{ij'}})$  for all  $j \in \{1, \dots, m_i\}$  and  $j' \in \{1, \dots, n_i\}$ .

**Remark 2.2.1** 1. If  $k = k_\alpha$  (in particular, if  $k$  is real closed), then  $m_i = 1$ .

2. By Proposition 1.6.5,  $\sum_{j=1}^{m_i} b_{ji} \nu_\alpha \mathbf{Q}^{\theta_{ji}} \neq 0$ .

**Conjecture 2.2.2** 1. Let

$$C = \left\{ \delta \in \text{Sper}A \left| \begin{array}{ll} \nu_\delta(\mathbf{Q}^{\theta_{ji}}) < \nu_\delta(\mathbf{Q}^{\epsilon_{j'i}}) & \text{for all } j \in \{1, \dots, m_i\}, j' \in \{1, \dots, n_i\} \\ \text{sgn}_\delta(Q_q) = \text{sgn}_\alpha(Q_q) & \text{for all } Q_q \text{ appearing in } \mathbf{Q}^{\theta_{ji}} \\ \text{sgn}_\delta(\sum_{j=1}^{m_i} b_{ji} \mathbf{Q}^{\theta_{ji}}) = & \text{sgn}_\alpha(\sum_{j=1}^{m_i} b_{ji} \mathbf{Q}^{\theta_{ji}}) \end{array} \right. \right\}. \quad (91)$$

$C$  is connected.

2. Let  $C'$  defined by the inequalities

$$\left| \sum_{j=1}^{m_i} b_{ji} \mathbf{Q}^{\theta_{ji}} \right| >_\delta n_i |\mathbf{Q}^{\epsilon_{j'i}}| \quad \forall i \in \{1, \dots, r\}, \quad \forall j' \in \{1, \dots, n_i\} \quad (92)$$

and the two sign conditions appearing in (91).  $C'$  is connected.

**Remark 2.2.3** 1. We have  $\alpha, \beta \in C$ .

2.  $C \cap \{f_1 \cdots f_r = 0\} = \emptyset$ . Indeed, note that inequalities (91) imply that, for every  $\delta \in C$ ,  $f_i$  has the same sign as  $\sum_{j=1}^{m_i} b_{ji} \mathbf{Q}^{\theta_{ji}}$ ; in particular, none of the  $f_i$  vanish on  $C$ .

3. Either of those conjectures implies the Connectedness Conjecture.

## Part 3. A proof of the conjecture for arbitrary regular 2-dimensional rings

### 3.1 Approximate roots in dimension 2 and Zariski's theory

In the special case of regular 2-dimensional local rings, the theory of approximate roots is well known : see, for instance [45], Appendix 5 or [36]. We briefly recall the construction here since it is much simpler than in the general case.

We start with two purely combinatorial lemmas about semigroups. Take an integer  $g \geq 3$ .

**Lemma 3.1.1** Let  $\beta_1, \beta_2, \dots, \beta_g$  be positive elements in some ordered group. Let  $\alpha_j, j \in \{2, \dots, g\}$  be positive integers. Assume

$$\beta_i \geq \alpha_{i-1} \beta_{i-1}, \quad i \in \{3, \dots, g\}. \quad (93)$$

Let  $\gamma_1, \dots, \gamma_g$  be integers such that  $0 \leq \gamma_j < \alpha_j$  for  $2 \leq j \leq g$  and  $\sum_{j=1}^g \gamma_j \beta_j \geq \gamma_g \beta_g$ . Then  $\gamma_1 > 0$ .

Proof : We prove by descending induction that  $\sum_{j=1}^i \gamma_j \beta_j \geq \alpha_i \beta_i$  for  $i \geq 2$ . The case  $i = g$  is given by hypothesis. Assume then that  $\sum_{j=1}^{i+1} \gamma_j \beta_j \geq \alpha_{i+1} \beta_{i+1}$ . Subtracting  $\gamma_{i+1} \beta_{i+1}$  and using the fact that  $\gamma_{i+1} < \alpha_{i+1}$ , we obtain  $\sum_{j=1}^i \gamma_j \beta_j \geq (\alpha_{i+1} - \gamma_{i+1}) \beta_{i+1} \geq \alpha_i \beta_i$ . This completes the induction. So for  $i = 2$ , we obtain  $\gamma_1 \beta_1 + \gamma_2 \beta_2 \geq \alpha_2 \beta_2$ ; subtractiong  $\gamma_2 \beta_2$  and using the fact that  $\gamma_2 < \alpha_2$ , we get  $\gamma_1 \beta_1 \geq (\alpha_2 - \gamma_2) \beta_2 > 0$ , hence  $\gamma_1 > 0$ .  $\square$

**Notation.** Let  $\beta_1, \beta_2, \dots, \beta_g$  be positive elements in some ordered group. We will denote by  $(\beta_1, \dots, \beta_{i-1})$  the group generated by  $\beta_1, \dots, \beta_{i-1}$  and by  $sg(\beta_1, \dots, \beta_{i-1})$  the semigroup generated by  $\beta_1, \dots, \beta_{i-1}$ , that is, the semigroup formed by all the  $\mathbb{N}$ -linear combinations of  $\beta_1, \dots, \beta_{i-1}$ . For  $i \in \{2, \dots, g\}$ ,  $\alpha'_i$  will denote the smallest positive integer such that  $\alpha'_i \beta_i \in (\beta_1, \dots, \beta_{i-1})$ . If there is no such integer, we put  $\alpha'_i = \infty$ . Write

$$\alpha'_i \beta_i = \sum_{j=1}^{i-1} \alpha_{ji} \beta_j \quad \text{where } \alpha_{ji} \in \mathbb{Z}. \quad (94)$$

**Lemma 3.1.2** *Let  $\beta_1, \beta_2, \dots, \beta_g$  be positive rational numbers. Assume that for  $i \in \{3, \dots, g\}$  we have*

$$\{a \in (\beta_1, \dots, \beta_{i-1}) \mid a \geq \alpha'_{i-1} \beta_{i-1}\} = \{a \in sg(\beta_1, \dots, \beta_{i-1}) \mid a \geq \alpha'_{i-1} \beta_{i-1}\}; \quad (95)$$

*in other words,  $\alpha_{ji} \geq 0$  for all  $j \in \{1, \dots, i-1\}$  in (94). Then*

$$\{a \in (\beta_1, \dots, \beta_g) \mid a \geq \alpha'_g \beta_g\} = \{a \in sg(\beta_1, \dots, \beta_g) \mid a \geq \alpha'_g \beta_g\}. \quad (96)$$

**Proof.** We use induction on  $g$ . Multiplying all the  $\beta_i$  by the same rational number does not change the problem, so we may assume that  $\beta_1, \beta_2, \dots, \beta_g$  are positive integers, such that  $\gcd(\beta_1, \beta_2, \dots, \beta_g) = 1$ .

Suppose that  $g > 3$  and assume the Lemma for  $g-1$ . Write

$$\alpha'_g \beta_g = \sum_{j=1}^{g-1} \alpha_{jg} \beta_j. \quad (97)$$

To prove (96), let  $\beta = \gamma_1 \beta_1 + \gamma_2 \beta_2 + \dots + \gamma_g \beta_g$  be an element of  $\{a \in (\beta_1, \dots, \beta_g) \mid a \geq \alpha'_g \beta_g\}$ . Using the relation (97) we can write, for each  $n \in \mathbb{Z}$ ,

$$\beta = \sum_{j=1}^{g-1} (\gamma_j - n \alpha_{jg}) \beta_j + (\gamma_g + n \alpha'_g) \beta_g = \sum_{j=1}^{g-1} \gamma'_j \beta_j + (\gamma_g + n \alpha'_g) \beta_g.$$

After replacing  $\gamma_g$  by  $\gamma_g + n \alpha'_g$  for a suitable  $n$ , we may assume that  $0 \leq \gamma_g < \alpha'_g$ . Since  $\beta \geq \alpha'_g \beta_g$ , this implies that

$$\sum_{j=1}^{g-1} \gamma_j \beta_j \geq \beta_g > \alpha'_{g-1} \beta_{g-1}. \quad (98)$$

By the induction assumption, we may take  $\gamma_j \geq 0$  in (98). This completes the proof of the Lemma.  $\square$

**Corollary 3.1.3** *Let  $\beta_1, \dots, \beta_g$  be positive rational numbers satisfying*

$$\beta_i \geq \alpha'_{i-1} \beta_{i-1}, \quad i \in \{3, \dots, g\}. \quad (99)$$

*Then equalities (95) and (96) hold.*

**Proof :** For  $i = 3$ , (95) is immediate. Now the corollary follows from Lemma 3.1.2 by induction on  $i$ .  $\square$

Let  $\nu$  be a valuation centered at  $A$  and let  $(x, y)$  be a  $\nu$ -prepared system of coordinates, such that  $\nu(x) = \nu(\mathbf{m})$ . In what follows, we will omit the description of  $\mathcal{V}(\gamma), \Lambda(\gamma), \Theta(\gamma)$ , since in the simplified situation of  $n = 2$ , the sets  $\Psi(\gamma)$  suffice to carry out the entire construction.

Put  $Q_1 = x$ ,  $Ex(Q_1) = x$ ,  $Q_2 = y$ ,  $Ex(Q_2) = y$  and  $\beta_i = \nu(Q_i)$ ,  $i \in \{1, 2\}$ . Let  $\Psi(\beta_1) = \emptyset$ . For  $\gamma \in \Phi$ ,  $\beta_1 < \gamma < \beta_2$ ,  $\Psi(\gamma) = \{x\}$  and  $\Psi(\beta_2+) = \{x, y\}$ . Let  $k_1 = k$ ,  $k_2 = k \left( \frac{\text{in}_\nu(Q_2^{\alpha'_2})}{\text{in}_\nu(Q_1^{\alpha_{12}})} \right)$ .

If  $\beta_1, \beta_2$  are rationally independent, then  $\alpha'_2 = \infty$  and the construction stops, there are no more approximate roots. In this case, all the  $\nu$ -ideals are generated by monomial in  $(x, y)$ .

Assume then  $\alpha'_2 < \infty$ . This means that there is a relation  $\alpha'_2\beta_2 = \alpha_{12}\beta_1$  for a positive integer  $\alpha_{12}$ . We prove that (95) is satisfied for  $i = 3$ . Let  $\beta = \gamma_1\beta_1 + \gamma_2\beta_2$  be an element of

$$\{a \in (\beta_1, \beta_2) \mid a \geq \alpha'_2\beta_2\}.$$

As  $\alpha'_2\beta_2 = \alpha_{12}\beta_1$ , we have, for each  $n \in \mathbb{Z}$ ,  $\beta = (\gamma_1 - n\alpha_{12})\beta_1 + (\gamma_2 + n\alpha'_2)\beta_2$ . After replacing  $\gamma_2$  by  $\gamma_2 + n\alpha'_2$  for a suitable  $n$ , we may assume that  $0 \leq \gamma_2 < \alpha'_2$ . Since  $\beta \geq \alpha'_2\beta_2$ , this implies that  $\gamma_1 \geq 0$ .

Then we have  $\nu(Q_2^{\alpha'_2}) = \nu(Q_1^{\alpha_{12}})$ , hence the image of  $\frac{\text{in}_\nu(Q_2^{\alpha'_2})}{\text{in}_\nu(Q_1^{\alpha_{12}})}$  in  $k_\nu$  is not zero. If  $\frac{\text{in}_\nu(Q_2^{\alpha'_2})}{\text{in}_\nu(Q_1^{\alpha_{12}})}$  is algebraic over  $k$ , this means that it satisfies an algebraic equation of the form

$$X^d + \bar{a}_1 X^{d-1} + \dots + \bar{a}_d = 0, \quad \bar{a}_i \in k. \quad (100)$$

Let  $a_i$  be a representative of  $\bar{a}_i$  in  $A$ . Let  $\alpha_2 = d\alpha'_2$  and

$$Q_3^{(1)} = Q_2^{\alpha_2} + \sum_{\ell=1}^d a_\ell Q_2^{\alpha'_2(d-\ell)} Q_1^{\alpha_{12}\ell}. \quad (101)$$

The expression  $Ex(Q_3^{(1)})$  is just the right hand side of this formula.

Let  $\beta_3^{(1)} = \nu(Q_3^{(1)})$ , then  $\beta_3^{(1)} > \nu(Q_2^{\alpha_2}) = \alpha_2\beta_2 \geq \alpha'_2\beta_2$  and the elements  $(\beta_1, \beta_2, \beta_3^{(1)})$  satisfy the conclusion of Lemma 3.1.2.

By construction,  $\alpha_2\beta_2$  is the smallest element of  $\Phi$  such that the monomials

$$\{Q_1^{\gamma_1} Q_2^{\gamma_2} \mid \nu(Q_1^{\gamma_1} Q_2^{\gamma_2}) = \gamma_1\beta_1 + \gamma_2\beta_2 = \alpha_2\beta_2\}$$

are  $k$ -linearly dependent. The unique  $k$ -linear dependence relation is given by  $Q_3^{(1)}$ . Hence, according to the general construction of §2, we have  $\Theta(\beta) = \{Q_3^{(1)}\}$  for  $\alpha_2\beta_2 \geq \beta \geq \beta_3^{(1)}$  and  $\Psi(\beta_3^{(1)}+) = \{Q_1, Q_2, Q_3^{(1)}\}$ .

Assume that  $i \geq 3$  and that elements  $Q_1, \dots, Q_{i-1}, Q_i^{(j)}$  are already defined. Let

$$\beta_q = \nu(Q_q), \quad (102)$$

$$\beta_i^{(j)} = \nu(Q_i^{(j)}). \quad (103)$$

Assume that the initial form  $\text{in}_\nu Q_q$  is algebraic over  $k[\text{in}_\nu Q_1, \dots, \text{in}_\nu Q_{q-1}]$  for  $q \in \{2, \dots, i-1\}$ . Let  $\alpha_q$  denote the degree of its minimal polynomial. Note that, in particular, all of  $\beta_2, \dots, \beta_{i-1}$  are rational multiples of  $\beta_1$ . Assume that  $\beta_q > \alpha_{q-1}\beta_{q-1}$ ,  $q \in \{3, \dots, i-1\}$  and  $\beta_i^{(j)} > \alpha_{i-1}\beta_{i-1}$ . Assume that, in the notation of §1.3, we have  $\Psi(\beta_i^{(j)}+) = \{Q_1, \dots, Q_{i-1}, Q_i^{(j)}\}$ .

A monomial  $\prod_{\ell=1}^{i-1} Q_\ell^{\epsilon_\ell}$  is **standard** if

$$0 \leq \epsilon_\ell < \alpha_\ell \text{ for } \ell \in \{2, \dots, i-1\}. \quad (104)$$

This allows us to extend the notion of standard to monomials with  $\epsilon_1 < 0$  : such a monomial is called standard if (104) is satisfied. Similarly, we may talk about standard monomials in  $\text{in}_\nu Q_1, \dots, \text{in}_\nu Q_{i-1}$ .

Assume, in addition, that we have defined elements  $z_2, \dots, z_{i-1} \in k_\nu$ , algebraic over  $k$ , where  $z_\ell$  is a  $k$ -linear combination of standard monomials in  $\text{in}_\nu Q_1, \dots, \text{in}_\nu Q_\ell$  of degree 0. Let  $k_\ell = k(z_2, \dots, z_\ell)$ . We obtain a tower of finite field extensions  $k = k_1 \subset k_2 \cdots \subset k_{i-1} \subset k_\nu$ .

If  $\text{in}_\nu Q a_i^{(j)}$  is transcendental over  $k[\text{in}_\nu Q_1, \dots, \text{in}_\nu Q_{i-1}]$ , put  $Q_i = Q_i^{(j)}$  and the construction stops.

Assume  $\text{in}_\nu Q a_i^{(j)}$  is algebraic over  $k[\text{in}_\nu Q_1, \dots, \text{in}_\nu Q_{i-1}]$ . Then  $\beta_i^{(j)} \in \sum_{q=1}^{i-1} \mathbb{Q} \beta_q$ . Let  $\alpha_i^{(j)}$  be the smallest positive integer such that  $\alpha_i^{(j)} \beta_i^{(j)} \in (\beta_1, \dots, \beta_{i-1})$ .

Then  $\nu \left( \left( Q_i^{(j)} \right)^{\alpha_i^{(j)}} \right) = \nu \left( \prod_{r=1}^{i-1} Q_r^{\alpha_{ri}^{(j)}} \right)$ , hence the image of  $\frac{\text{in}_\nu \left( Q_i^{(j)} \right)^{\alpha_i^{(j)}}}{\text{in}_\nu \prod_{r=1}^{i-1} Q_r^{\alpha_{ri}^{(j)}}}$  in  $k_\nu$  is not zero. By Lemma 3.1.2, we may take  $\alpha_{ri} \geq 0$  for  $1 \leq r \leq i-1$ .

The assumption on  $\text{in}_\nu Q_i^{(j)}$  implies that  $\frac{\text{in}_\nu \left( Q_i^{(j)} \right)^{\alpha_i^{(j)}}}{\text{in}_\nu \prod_{r=1}^{i-1} Q_r^{\alpha_{ri}^{(j)}}}$  satisfies an algebraic equation of the form

$$X^d + \bar{a}_1 X^{d-1} + \cdots + \bar{a}_d = 0, \quad \bar{a}_\ell \in k_{i-1}. \quad (105)$$

For  $\ell \in \{1, \dots, d\}$ , write

$$\bar{a}_\ell \left( \prod_{r=1}^{i-1} \text{in}_\nu Q_r^{\alpha_{ri}^{(j)}} \right)^\ell = \sum_{\gamma=(\gamma_1, \dots, \gamma_{i-1})} \bar{b}_{\ell\gamma} \prod_{r=1}^{i-1} \text{in}_\nu Q_r^{\gamma_r}$$

as a  $k$ -linear combination of standard monomials. By Corollary 3.1.3, we have  $\gamma_1 \geq 0$  whenever  $\bar{b}_{\ell\gamma} \neq 0$ .

Let  $b_{\ell\gamma}$  be a representative of  $\bar{b}_{\ell\gamma}$  in  $A$ . Let  $\alpha_i^{(j)} = d\alpha_i^{\prime(j)}$  and

$$Q = \left( Q_i^{(j)} \right)^{\alpha_i^{(j)}} + \sum_{\ell=1}^d \left( \sum_{\gamma=(\gamma_1, \dots, \gamma_{i-1})} b_{\ell\gamma} \prod_{r=1}^{i-1} Q_r^{\gamma_r} \right) Q_i^{\alpha_i^{\prime(j)}(d-\ell)}. \quad (106)$$

Then

$$\nu(Q) > \nu \left( \left( Q_i^{(j)} \right)^{\alpha_i^{(j)}} \right) = \alpha_i^{(j)} \beta_i^{(j)} \geq \alpha_i^{\prime(j)} \beta_i^{(j)} > \alpha_{i-1} \beta_{i-1} \geq \alpha_{i-1}^{\prime} \beta_{i-1}. \quad (107)$$

If  $\text{in}_\nu Q_i^{(j)} \notin k[\text{in}_\nu Q_1, \dots, \text{in}_\nu Q_{i-1}]$  (which is equivalent to saying that  $\alpha_i^{(j)} > 1$ ), put  $Q_i = Q_i^{(j)}$ ,  $Ex(Q_i) = Ex(Q_i^{(j)})$ ,  $\beta_i = \beta_i^{(j)}$ ,  $\alpha_i = \alpha_i^{(j)}$ ,  $\alpha_i^{\prime} = \alpha_i^{\prime(j)}$ ,  $Q_{i+1}^{(1)} = Q$ ,  $\beta_{i+1}^{(1)} = \nu \left( Q_{i+1}^{(1)} \right)$ .

Formulae (106) and (107) become

$$Q_{i+1}^{(1)} = Q_i^{\alpha_i} + \sum_{\ell=1}^d \left( \sum_{\gamma=(\gamma_1, \dots, \gamma_{i-1})} b_{\ell\gamma} \prod_{r=1}^{i-1} Q_r^{\gamma_r} \right) Q_i^{\alpha_i^{\prime}(d-\ell)}. \quad (108)$$

and

$$\beta_{i+1}^{(1)} = \nu \left( Q_{i+1}^{(1)} \right) > \nu(Q_i^{\alpha_i}) = \alpha_i \beta_i \geq \alpha_i^{\prime} \beta_i. \quad (109)$$

The expression  $Ex \left( Q_{i+1}^{(1)} \right)$  is just the right hand side of (108).

For  $\beta_i^{(j)} < \gamma \leq \beta_{i+1}^{(1)}$ , we have  $\Psi(\gamma) = \Psi \left( \beta_i^{(j)} + \right)$  and  $\Psi \left( \beta_{i+1}^{(1)} + \right) = \{Q_1, \dots, Q_i, Q_{i+1}^{(1)}\}$ . Moreover, the elements  $(\beta_1, \dots, \beta_{i+1}^{(1)})$  satisfy the conclusion of Lemma 3.1.2.

If  $\text{in}_\nu Q_i^{(j)} \in k[\text{in}_\nu Q_1, \dots, \text{in}_\nu Q_{i-1}]$  (which is equivalent to saying that  $\alpha_i^{(j)} = 1$ ), put  $Q_i^{(j+1)} = Q$  and  $\beta_i^{(j+1)} = \nu \left( Q_i^{(j+1)} \right)$ .

Formulae (106) and (107) become

$$Q_i^{(j+1)} = Q_i^{(j)} + \sum_{\gamma=(\gamma_1, \dots, \gamma_{i-1})} b_{1\gamma} \prod_{r=1}^{i-1} Q_r^{\gamma_r} \quad (110)$$

and

$$\beta_i^{(j+1)} = \nu \left( Q_i^{(j+1)} \right) > \beta_i^{(j)} > \alpha_{i-1}^i \beta_{i-1}. \quad (111)$$

The expression  $Ex \left( Q_i^{(j+1)} \right)$  is just the right hand side of (110).

For  $\beta_i^{(j)} < \gamma \leq \beta_i^{(j+1)}$ , we have  $\Psi(\gamma) = \Psi \left( \beta_i^{(j)} + \right)$  and  $\Psi \left( \beta_i^{(j+1)} + \right) = \{Q_1, \dots, Q_{i-1}, Q_i^{(j+1)}\}$ . Moreover, the elements  $(\beta_1, \dots, \beta_{i-1}, \beta_i^{(j+1)})$  satisfy the conclusion of Lemma 3.1.2.

**Remark 3.1.4** *Either the process stops after a finite number of steps or we obtain an infinite sequence*

$$\mathbf{Q} = Q_1, Q_2, \dots, Q_i, \dots \quad (112)$$

or a sequence

$$\mathbf{Q} = Q_1, Q_2, \dots, Q_{i-1}, Q_i^{(j)}, \quad j \in \mathbb{N}. \quad (113)$$

In the case when  $\mathbf{Q}$  is given by (112), it is a system of approximate roots, whether or not  $A$  is complete. In the case (113) assume, in addition, that the ring  $A$  is  $\mathfrak{m}$ -adically complete. In that case,

$$Q_\infty = \lim_{i \rightarrow \infty} Q_i$$

is a well defined element of  $A$  and  $\mathbf{Q} \cup \{Q_\infty\}$  is a system of approximate roots.

### 3.2 Real geometric surfaces and their blowings up

Let  $A$  be a ring and  $U$  an open subset of  $\text{Sper}(A)$ . Let  $S_U$  denote the multiplicative set  $S_U = \{g \in A \mid g(\alpha) \neq 0 \text{ for all } \alpha \in U\}$ . Let  $A_U = A_{S_U}$ . We have a natural ring homomorphism

$$\rho_U : A_U \rightarrow \prod_{\alpha \in U} A(\alpha).$$

Define the ring  $O_U$  to be the ring of all maps

$$f : U \rightarrow \prod_{\alpha \in U} A(\alpha)$$

satisfying the following conditions :

- (1)  $\forall \alpha \in U, f(\alpha) \in A(\alpha)$ ;

(2) there exists an open covering  $U = \cup_i U_i$  and, for each  $i$ , an element  $f_i \in A_{U_i}$  such that  $\forall \beta \in U_i$ , we have  $\rho_{U_i}(f_i)_\beta = f(\beta)$ .

The functor which sends  $U$  to  $O_U$  makes  $\text{Sper}(A)$  into a locally ringed space which is called *affine real geometric space*. This notion is inspired by the notion of real closed spaces defined by Niels Schwartz ([35]).

**Remark 3.2.1** Note that  $A_U \hookrightarrow O_U$  and, if  $U$  is connected, this becomes an equality.

**Definition 3.2.2** A real geometric space is a locally ringed space  $(X, O_X)$  which admits an open covering  $X = \bigcup_{i=1}^s \text{Sper}(A_i)$  such that each  $(U_i, O_X|_{U_i})$  is isomorphic (as locally ringed space) to an affine real geometric space.

**Definition 3.2.3** A real geometric surface is a real geometric space  $X$  where all  $A_i$  can be chosen to be regular 2-dimensional noetherian rings.

**Definition 3.2.4** A marked real geometric surface is a real geometric surface  $X$  together with the following additional data :

(1) A finite covering  $X = \bigcup_{i=1}^s \text{Sper}(A_i)$  where each  $A_i$  is a regular 2-dimensional noetherian ring.

(2) For each  $i$  and each prime ideal of height two  $\mathfrak{p} \subset A_i$ , a regular system of parameters at the center of  $(A_i)_\mathfrak{p}$ .

Let  $A$  be a regular 2-dimensional ring,  $\mathfrak{m}$  a maximal ideal of  $A$  and  $(x, y)$  a regular system of parameters of  $A_\mathfrak{m}$ .

**Definition 3.2.5** A **blowing up** of  $\text{Sper} A$  along  $\mathfrak{m}$  is the marked real geometric surface  $X'$  defined as follows.  $X' = \text{Sper} A'_1 \cup \text{Sper} A'_2$ , where  $A'_1 = A \left[ \frac{y}{x} \right]$ ,  $A'_2 = A \left[ \frac{x}{y} \right]$  and

$$\text{Sper} A'_1 \cap \text{Sper} A'_2 = \text{Sper} A'_1 \setminus \{y = 0\} = \text{Sper} A'_2 \setminus \{x = 0\}.$$

We have a natural surjective morphism  $\pi : X' \rightarrow \text{Sper} A$ .

Take a prime ideal  $\mathfrak{p} \subset \text{Sper} A'_1$  such that  $\text{ht}(\mathfrak{p}) = 2$ . Then  $\mathfrak{p} \cap A = \{\mathfrak{m}\}$ . The field extension  $k \hookrightarrow \frac{A'_1}{\mathfrak{p}}$  is algebraic. Let  $\bar{y}_1$  be the image of  $\frac{y}{x}$  in  $\frac{A'_1}{\mathfrak{p}}$  and let  $f = z^d + \bar{a}_1 z^{d-1} + \dots + \bar{a}_d \in k[z]$  be the monic polynomial of  $\bar{y}_1$  over  $k$ . Let  $a_i$  be a representative of  $\bar{a}_i$  in  $A$ . Put  $x_1 = x$ ,  $y_1 = \left(\frac{y}{x}\right)^d + a_1 \left(\frac{y}{x}\right)^{d-1} + \dots + a_d \in A'_1$ . And similarly for  $A'_2$ .

If  $\eta$  is a point of  $\text{Sper} A$  with  $\mathfrak{m} = \mathfrak{p}_\eta$ , we will also say that  $\pi : X' \rightarrow \text{Sper} A$  is a **blowing up along**  $\eta$ .

**Definition 3.2.6** Let  $\delta$  be a point of  $\text{Sper} A$  and let  $\pi : A \rightarrow A'$  be a local blowing up and suppose  $\mathfrak{p}(\delta) \not\subseteq \mathfrak{m}$ . The **strict transform**  $\delta'$  of  $\delta$  is defined as follows. Let  $\mathfrak{p}_{\delta'}$  be the strict transform of  $\mathfrak{p}_\delta$  and  $\leq_{\delta'}$  be the order of  $\kappa(\mathfrak{p}_{\delta'})$  induced by  $\leq_\delta$  via the natural isomorphism  $\kappa(\mathfrak{p}_\delta) \cong \kappa(\mathfrak{p}_{\delta'})$ .

**Remark 3.2.7** Fix an order on  $k$ . Let  $D$  be the set of points  $\delta \in \text{Sper}(k[z])$  which induces the given order on  $k$ . Given 2 points  $\delta_1 \neq \delta_2 \in D$  such that  $\text{ht}(\mathfrak{p}_{\delta_i}) = 1$ , we view  $\delta_1, \delta_2$  as elements of  $\bar{k}$ . We may speak about the interval  $(\delta_1, \delta_2) = \{\delta \in E \mid \delta_1 < z(\delta) < \delta_2\}$ . If  $\delta$  is not a closed point, we compare  $\delta_i$  and  $z(\delta)$  via the natural embeddings  $k[z](\delta) \hookrightarrow \bar{k}(z)$  and  $\bar{k} \subset \bar{k}(z)$ .



Now, given a blowing up along  $\mathfrak{m}$  as above, consider the open set  $\text{Sper}(A[\frac{y}{x}])$ . The set of points  $\delta \in \text{Sper}(A[\frac{y}{x}])$  such that  $x(\delta) = 0$  and which induce the given order on  $k$  is homeomorphic to  $E$ .

Finally, let  $X$  be a real algebraic surface such that  $D \subset \text{Sper}k[z] \subset X$ . Let  $+\infty$  denote the point of  $D$  with support  $(0)$  such that  $z(+\infty) > c$  for all  $c \in k$ . Let  $\overline{\infty}$  be the closed point of  $X$  such that  $\overline{\infty} \in \overline{\{+\infty\}}$ . Assume that there is an open set  $\text{Sper} A_i \subset X$  such that  $\mathfrak{p}_{\overline{\infty}}$  in  $A_i$  has height 2. We extend the above notion of interval to include the case when  $\delta_2 = \overline{\infty}$  with the obvious meaning assigned to  $[\delta_1, \overline{\infty}] = \bigcup_{\delta > \delta_1} [\delta_1, \delta] \cup \{\overline{\infty}\}$ ,  $(\delta_1, \overline{\infty}), \dots$ . Similarly, we may take a closed point  $-\overline{\infty} \in \overline{\{-\infty\}}$  and talk about intervals of the form  $(\overline{\infty}, \delta_1)$ ,  $[\overline{\infty}, \delta_1)$  and so on.

**Lemma 3.2.8** *Let  $E$  be as in the remark before and  $\delta_1 < \delta_2 \subset E$  such that  $ht(\mathfrak{p}_{\delta_i}) = 1$ . The semi-open interval  $[\delta_1, \delta_2)$  and the open interval  $(\delta_1, \delta_2)$  are connected.*

Proof : We will prove it for the open case, the semi-open being similar. Let  $k \hookrightarrow \overline{k}$  be the inclusion of  $k$  into its real closure determined by the given order. This map corresponds to a morphism  $\text{Sper}(\overline{k}[z]) \rightarrow \text{Sper}(k[z])$  which induces a homeomorphism between  $E$  and  $\text{Sper}(\overline{k}[z])$  sending  $(\delta_1, \delta_2)$  to an interval  $(\overline{\delta}_1, \overline{\delta}_2)$  where  $\overline{\delta}_1, \overline{\delta}_2 \in \overline{k}$ . It is well-known and easy to prove that such an interval is connected - in the spectral topology (see for instance [7]).  $\square$

**Definition 3.2.9** *Let  $\mathcal{E} = \{\epsilon \in \text{Sper}(A) \mid \mathfrak{p}_\epsilon = \mathfrak{m}\}$ . The sets  $\pi^{-1}(\epsilon)$ ,  $\epsilon \in \mathcal{E}$  are called the **components** of  $\pi^{-1}(\mathfrak{m})$ .*

Let  $X = \bigcup_{i=1}^s \text{Sper}(A_i)$  be a marked real geometric surface and  $\mathfrak{m}_1$  a maximal ideal of  $A_1$  such that there exists  $\delta \in \text{Sper} A_1$  with  $\mathfrak{p}_\delta = \mathfrak{m}_1$ . For  $1 \leq i \leq s$  such that  $\delta \in \text{Sper} A_i$ , let  $\mathfrak{m}_i$  denote the support of  $\delta$  in  $A_i$ .

If  $\delta \notin \text{Sper} A_i$ , let  $X'_i = \text{Sper} A_i$  with the identity map  $X'_i \rightarrow \text{Sper} A_i$ . If  $\delta \in \text{Sper} A_i$ , let  $X'_i \rightarrow \text{Sper} A_i$  be the blowing up of  $\text{Sper} A_i$  along  $\mathfrak{m}_i$ . Let  $\tilde{x}, \tilde{y}$  be a regular system of parameters at  $\mathfrak{m}_i$ , then

$$X'_i = \text{Sper} A_i \left[ \frac{\tilde{x}}{\tilde{y}} \right] \cup A_i \left[ \frac{\tilde{y}}{\tilde{x}} \right].$$

The marked real geometric surfaces  $X'_1, \dots, X'_s$  and the maps  $X'_i \rightarrow \text{Sper} A_i$  glue together in a natural way to give a marked real geometric surface  $X' = \bigcup_{i=1}^s X'_i$  and the map  $X' \rightarrow X$ , which we call **the blowing up of  $X$  along  $\mathfrak{m}_1$**  or the point blowing up of  $X$  along  $\mathfrak{m}_1$ .

Let  $(A, \mathfrak{m}, k)$  be a regular 2-dimensional local ring and  $(x, y)$  a regular system of parameter. Now consider a sequence

$$X_t \xrightarrow{\pi_{t-1}} \dots \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_0} \text{Sper}(A) \tag{114}$$

of point blowings up where the first blowing up  $\pi_0 : X_1 \rightarrow \text{Sper}(A)$  is the blowing up along  $\mathfrak{m}$ .

Fix a point  $\epsilon \in \text{Sper}(A)$  such that  $\mathfrak{p}_\epsilon = \mathfrak{m}$  - this is equivalent to fixing a total ordering on  $k$ . For  $q \in \{0, \dots, t-1\}$ , let  $\eta_q \in X_q$  be the closed point, compatible with the given order, such that  $\pi_q$  is a blowing up along  $\eta_q$ .

For  $i \in \{1, \dots, t\}$ , let  $X_i = \bigcup_{j=1}^{s_i} \text{Sper}(A_{j_i})$  be the open affine covering in the definition of marked real geometric surface.

Let  $\rho_i = \pi_0 \circ \dots \circ \pi_{i-1} : X_i \rightarrow \text{Sper}(A)$ .

**Remark 3.2.10** *The real geometric space  $\rho_i^{-1}(\mathfrak{m})$  has the form  $\rho_i^{-1}(\mathfrak{m}) = \bigcup_\ell \text{Sper} B_{i\ell}$  with  $B_{i\ell} \cong k_{i\ell}[z_{i\ell}]$  where  $k_{i\ell}$  is a finite algebraic extension of  $k$  and  $z_{i\ell}$  is an independant variable.*

**Definition 3.2.11** A subset  $E \subset \rho_i^{-1}(\epsilon)$  is a component of  $\rho_i^{-1}(\epsilon)$  if  $E$  is either a component of  $\pi_{i-1}^{-1}(\eta_{i-1})$  or a strict transform of a component of  $\rho_{i-1}^{-1}(\epsilon)$  when  $i > 1$ .

**Definition 3.2.12** Let  $\rho_i : X_i \rightarrow \text{Sper}(A)$ . Fix a component  $E \subset \rho_i^{-1}(\epsilon)$ . Fix an index  $j \in \{1, \dots, s_i\}$ . A  $j$ -distinguished point of  $E$  is a point  $\delta \in E$  such that either  $\delta \notin \text{Sper}A_{ji}$  or  $\rho_i^{-1}(\{xy = 0\}) \supset \{x'y' = 0\}$  where  $(x', y')$  is a suitable regular system of parameters at  $\delta$ .

A  $j$ -maximal interval  $I$  is a subset  $I \subset E$  such that there exists  $j$ -distinguished points  $\delta_1, \delta_2 \in E$ ,  $\delta_1 \neq \delta_2$ , such that

- (1)  $I = [\delta_1, \delta_2]$
- (2) There are no  $j$ -distinguished points in  $I \setminus \{\delta_1, \delta_2\}$ .

**Remark 3.2.13** Note that a  $j$ -maximal interval may contain a  $\tilde{j}$ -distinguished point, where  $j \neq \tilde{j}$ .

**Proposition 3.2.14** Fix a component  $E \subset \rho_i^{-1}(\epsilon)$  and a maximal interval  $[\delta_1, \delta_2] \subset E$ . Take  $q \in \{1, 2\}$ . There exist  $j \in \{1, \dots, s_i\}$  and  $x_i, y_i \in A_{ji}$  such that

- (1) <sub>$i$</sub>   $[\delta_1, \delta_2] \setminus \{\delta_q\} \subset \text{Sper}A_{ji}$ ,
- (2) <sub>$i$</sub>  for all closed  $\delta \in [\delta_1, \delta_2] \setminus \{\delta_q\}$ ,  $x_i$  is a part of regular system of parameters of  $(A_{ji})_{\mathfrak{p}_\delta}$ ,
- (3) <sub>$i$</sub>   $[\delta_1, \delta_2] \cap \text{Sper}A_{ji} = \{\eta \in \text{Sper}A_{ji} \mid x_i(\eta) = 0 \text{ and } \bar{\delta}_1 \leq y_i(\eta) \leq \bar{\delta}_2\}$  where  $\bar{\delta}_1, \bar{\delta}_2 \in \overline{k_{ji}} \cup \{-\infty, \infty\}$  where the notations are as in the Remark 3.2.7 and the proof of Lemma 3.2.8.

Proof : Let  $i = 1$ . We have  $X_1 = \text{Sper}A[\frac{y}{x}] \cup \text{Sper}A[\frac{x}{y}]$ . Denote  $A[\frac{y}{x}]$  by  $A_{11}$  and  $A[\frac{x}{y}]$  by  $A_{21}$ . Let  $x_1 = x, y_1 = \frac{y}{x}$ . Fixing the component  $E$  is equivalent to fixing a global order on  $k$ ; this data is already given. We have  $E \cap \text{Sper}A[y_1] \subset \text{Sper}k[y_1]$ . Let the notation be as in Remark 3.2.7 with  $y_1$  playing the role of  $z$ .

There are exactly two maximal intervals  $[0, \infty]$  and  $[\infty, 0]$ . Say, for example,  $I = [0, \infty]$ ,  $q = 2$ , then  $j = 1$  and  $(x_1, y_1)$  satisfy the conclusion of the Proposition. And similarly for the other three cases.

Now suppose the result true for  $i - 1$ . Let  $\delta_{k,i-1} = \pi_{i-1}(\delta_{k,i})$ ,  $k = 1, 2$ . Note that if  $[\delta_{1,i}, \delta_{2,i}]$  is a maximal interval of  $E$ , then  $[\delta_{1,i-1}, \delta_{2,i-1}]$  is a maximal interval of  $\pi_{i-1}(E)$ . So, by induction hypothesis, the results of the proposition hold for  $[\delta_{1,i-1}, \delta_{2,i-1}] \subset \pi_{i-1}(E)$ . Let  $\eta_{i-1}$  be the center of the blowing up  $\pi_{i-1}$ .

If  $\eta_{i-1} \notin \text{Sper}A_{j_{i-1}}$ , take  $j' \in \{1, \dots, s_i\}$  such that  $A_{j'_i} = A_{j_{i-1}}$ . Put  $x_i = x_{i-1}, y_i = y_{i-1}$ . Next assume that  $\eta_{i-1} \in \text{Sper}A_{j_{i-1}}$ . Let  $\tilde{x}_{i-1}, \tilde{y}_{i-1}$  be a system of regular parameters at  $\eta_{i-1}$  given the induction hypothesis.

If  $\eta_{i-1} \notin \pi_{i-1}(E)$ , take  $j'$  such that  $A_{j'_i} = A_{j_{i-1}}[\frac{\tilde{y}_{i-1}}{\tilde{x}_{i-1}}]$ . Put  $x_i = x_{i-1}, y_i = y_{i-1}$ .

From now on, assume that  $\eta_{i-1} \in \pi_{i-1}(E) \cap \text{Sper}A_{j_{i-1}}$  and  $x_{i-1} = \tilde{x}_{i-1}$ . Let  $j' \in \{1, \dots, s_i\}$  be such that  $A_{j'_i} = A_{j_{i-1}}[\frac{x_{i-1}}{y_{i-1}}]$ . Put  $x_i = \frac{x_{i-1}}{y_{i-1}}$  and  $y_i = y_{i-1}$ .  $\square$

We recall some basic facts from Zariski's theory of complete ideals in regular two-dimensional local rings.

Let  $(A, \mathfrak{m})$  be a regular 2-dimensional local ring,  $x, y$  a regular system of parameters and let  $\nu$  be a valuation centered at  $A$ .

**Definition 3.2.15** An ideal  $\mathcal{I}$  in a normal ring  $B$  is said to be integrally closed or complete if it contains all the elements  $x$  of  $B$  satisfying a monic equation of the form

$$x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0$$

where  $a_{n-i} \in \mathcal{I}^{n-i}$ .

An ideal  $\mathcal{I}$  in  $A$  is said to be simple if it cannot be factored in a non trivial way as a product of two other ideals.

A local blowing up of  $A$  with respect to  $\nu$  along  $\mathfrak{m}$  is a composition of an affine blowing up along  $\mathfrak{m}$  with respect to  $\nu$  with a localization at the center of  $\nu$ .

**Remark 3.2.16** Any  $\nu$ -ideal is a complete ideal.

Now let  $I$  be a simple  $\mathfrak{m}$ -primary  $\nu$ -ideal. Then

(1) The set

$$\mathfrak{m} = I_0 \supset I_1 \supset \cdots \supset I_\ell = I$$

is entirely determined by  $I$  (it does not depend on  $\nu$ ).

(2) Let  $A \rightarrow A_1$  be the local blowing up of  $\mathfrak{m}$  and, for  $i \geq 1$ , let  $I'_i$  be the transform of  $I_i$  (that is,  $I'_i = x^{-\mu} I_i A_1$  with  $\mu = \text{ord}_{\mathfrak{m}} I_i$ ). Then

$$\mathfrak{m}_1 = I'_1 \supset I'_2 \supset \cdots \supset I'_{\ell-1} = I'$$

is the set of simple  $\nu$ -ideals of  $A_1$  containing  $I'$ .

(3) Iterating this procedure  $\ell$ -times, we obtain a sequence of local blowing ups

$$(A, \mathfrak{m}) \rightarrow (A_1, \mathfrak{m}_1) \rightarrow \cdots \rightarrow (A_\ell, \mathfrak{m}_\ell) \tag{115}$$

such that the transform  $I^{(\ell)}$  of  $I$  is  $\mathfrak{m}_\ell$ . For any  $f \in A \setminus I$ , the strict transform of  $f$  in  $A_\ell$  is a unit of  $A_\ell$ .

We recall the following general result from the theory of approximate roots in regular 2-dimensional local rings ([36]).

Let  $A$  be a 2-dimensional regular local ring,  $\nu$  a valuation on  $A$ . Now let  $Q_k, k = 1, \dots, g+1$  be the approximate roots of  $\nu$  such that  $Q_1, \dots, Q_g \notin I$  and  $Q_{g+1} \in I$ . Each  $I_i$  is generated by the generalized monomials  $\prod Q_j^{\gamma_j}, \gamma_j \in \mathbb{N}$ , such that  $\sum \gamma_j \beta_j \geq \nu(I_i)$ .

**Proposition 3.2.17** For each  $i$ , there exist regular parameters  $x_i, y_i$  and natural numbers  $\ell_1 < \ell_2 < \cdots < \ell_g \leq \ell$  having the following properties :

- (1)  $x_{\ell_i}$  is a monomial of the form  $\prod_{j=1}^{i-1} Q_j^{\gamma_j}, \gamma_j \in \mathbb{N}$ ,
- (2)  $y_{\ell_i}$  is the strict transform of  $Q_i$  in  $A_{\ell_i}$ ,
- (3)  $Q_1, \dots, Q_{i-1}$  are monomials in  $x_{\ell_i}, y_{\ell_i}$  times a unit of  $(A_{\ell_i})_{(x_{\ell_i}, y_{\ell_i})}$ .

For  $\alpha, \beta \in \text{Sper} A$ , let  $Q_1, \dots, Q_s$  be the approximate roots common to  $\alpha$  and  $\beta$ .

**Corollary 3.2.18** If  $i \leq s$ , both  $\nu_\alpha$  and  $\nu_\beta$  are centered at  $(x_i, y_i)$ .

### 3.3 A proof of the Pierce-Birkhoff conjecture for regular 2-dimensional rings

Let  $A$  be a regular 2-dimensional ring. In this section, we prove that  $A$  is a Pierce-Birkhoff ring ([26]). Our proof is based on Madden's unpublished preprint ([27]), but there are some differences. Here, we have tried to present a proof which should be a pattern for a general proof of the conjecture in any dimension.

**Theorem 3.3.1** Let  $A$  be a 2-dimensional regular ring, then  $A$  is a Pierce-Birkhoff ring.

Actually, we prove that  $A$  satisfies the Definable Connectedness Conjecture and also, in the special case where  $A$  is excellent, the Connectedness Conjecture.

We start with some results which do not assume that  $A$  is excellent and which are needed in the proof of both of the above versions of the Connectedness Conjecture. Let  $\alpha, \beta \in \text{Sper}A$ . By Remark 0.1.10, we may assume that neither of  $\alpha, \beta$  is a specialization of the other.

There are two possibilities : either  $ht(\langle \alpha, \beta \rangle) = 1$  or  $ht(\langle \alpha, \beta \rangle) = 2$ .

### 3.3.1 The case of height 1

Let  $\delta$  be the most general common specialization of  $\alpha$  and  $\beta$  and let  $\mathfrak{p} = \sqrt{\langle \alpha, \beta \rangle}$  be the support of  $\delta$ . Then  $A_{\mathfrak{p}}$  is a discrete valuation ring; call  $t \in A$  a regular parameter. Since  $ht(\mathfrak{p}) = 1$  and neither of  $\alpha, \beta$  is a specialization of the other, we have  $\mathfrak{p}_{\alpha} = \mathfrak{p}_{\beta} = (0)$ . There are only two orders on  $A$  which induce the given order on  $A/\mathfrak{p}$  : one with  $t > 0$  and one with  $t < 0$ . Since  $\alpha \neq \beta$ ,  $\langle \alpha, \beta \rangle = \mathfrak{p}$  : of course, any generator  $g$  of  $\mathfrak{p}$  writes  $g = t^{\gamma} \frac{a}{b}$ ,  $a, b \notin \mathfrak{p}$ . As  $t \in \langle \alpha, \beta \rangle$ , if  $\gamma \geq 2$ ,  $\nu_{\alpha}(g) = \nu_{\beta}(g) > \nu_{\alpha}(t)$  so  $g \in \langle \alpha, \beta \rangle$  and if  $\gamma = 1$ ,  $g$  changes sign between  $\alpha$  and  $\beta$ , so again  $g \in \langle \alpha, \beta \rangle$ .

Now let  $f_1, \dots, f_r \notin \langle \alpha, \beta \rangle = \mathfrak{p}$ , so  $f_i(\delta) \neq 0$  for  $i \in \{1, \dots, r\}$ . As  $\delta \in \overline{\{\alpha\}}$  and  $\delta \in \overline{\{\beta\}}$ , we conclude that  $\alpha$  and  $\beta$  belong to the same connected component of  $\text{Sper}A \setminus \{f_1 \cdots f_r = 0\}$ .

### 3.3.2 The case of height 2

Now assume  $ht(\langle \alpha, \beta \rangle) = 2$ , that is  $\mathfrak{m} = \sqrt{\langle \alpha, \beta \rangle}$  is maximal. By Proposition 2.1.1, replacing  $A$  by  $A_{\mathfrak{m}}$  does not change the problem, so we may assume that  $A$  is local with maximal ideal  $\mathfrak{m}$ .

Let  $I = \langle \alpha, \beta \rangle$  and let  $Q_1, \dots, Q_{j+1}$  be as in the end of the preceding section.

**Lemma 3.3.2** For  $i \geq 3$ ,  $\nu_{\mathfrak{m}}(Q_i) = \prod_{j=1}^{i-1} \alpha_j$ .

Proof : Let  $i = 3$ , then we can write  $Q_3 = y^{\alpha_2} + \sum c_{rs} x^r y^s$  where  $c_{rs}$  is a unit in  $A$ , with  $\nu_{\alpha}(x^r y^s) \geq \alpha_2 \nu_{\alpha}(y)$ . As  $\nu_{\alpha}(y) \geq \nu_{\alpha}(x)$ , this implies  $\nu_{\mathfrak{m}}(Q_3) = \alpha_2$ .

Recall (cf. (106)) that

$$Q_{i+1} = Q_i^{\alpha_i} + \sum_{\ell=1}^d \left( \sum_{\gamma=(\gamma_1, \dots, \gamma_{i-1})} b_{\ell\gamma} \prod_{r=1}^{i-1} Q_r^{\gamma_r} \right) Q_i^{\alpha'_i(d-\ell)}.$$

Now to prove the lemma, it suffices to prove that

$$\alpha_i \nu_{\mathfrak{m}}(Q_i) \leq \nu_{\mathfrak{m}} \left( \left( \prod_{r=1}^{i-1} Q_r^{\gamma_r} \right) Q_i^{\alpha'_i(d-\ell)} \right) \quad (116)$$

for all  $\ell$  and  $\gamma$  such that  $b_{\ell\gamma} \neq 0$ .

First remark that, according to the inequalities (109) and (111), we deduce by an easy induction on  $i - \ell$  that

$$\frac{\beta_i}{\prod_{q=\ell}^{i-1} \alpha_q} \geq \beta_{\ell} \quad (117)$$

We have  $\alpha_i \beta_i = \sum_{j=1}^{i-1} \gamma_j \beta_j + \alpha'_i (d - \ell) \beta_i$ , so

$$\alpha'_i \ell \beta_i = \sum_{j=1}^{i-1} \gamma_j \beta_j \leq \sum_{j=1}^{i-1} \gamma_j \frac{\beta_i}{\prod_{q=j}^{i-1} \alpha_q}$$

by (117).

Dividing both sides by  $\frac{\beta_i}{\prod_{q=1}^{i-1} \alpha_q}$ , we get

$$\alpha'_i \ell \prod_{q=1}^{i-1} \alpha_q \leq \sum_{j=1}^{i-1} \gamma_j \prod_{q=1}^{j-1} \alpha_q. \quad (118)$$

By the induction assumption, the left hand side equals  $\nu_m(Q_i^{\alpha'_i \ell})$  and the right hand side equals  $\nu_m(\prod_{j=1}^{i-1} Q_j^{\gamma_j})$ . Therefore inequality (116) follows from inequality (118).  $\square$

Recall that in the special case of dimension 2, a monomial  $\prod_{j=1}^i Q_j^{\gamma_j}$  is standard if and only if  $\gamma_\ell < \alpha_\ell$ .

**Corollary 3.3.3** *Consider two standard monomials  $\prod_{j=1}^i Q_j^{\gamma_j}$  and  $\prod_{j=1}^i Q_j^{\gamma'_j}$  such that  $(\gamma_i, \gamma_{i-1}, \dots, \gamma_1) \leq_{lex} (\gamma'_i, \gamma'_{i-1}, \dots, \gamma'_1)$  and having the same  $\nu_\alpha$ -value. We have*

$$\nu_m \left( \prod_{j=1}^i Q_j^{\gamma_j} \right) \geq \nu_m \left( \prod_{j=1}^i Q_j^{\gamma'_j} \right).$$

Let  $\prod_{j=1}^i Q_j^{\gamma_j}$  and  $\prod_{j=1}^i Q_j^{\gamma'_j}$  be two monomials. We say that they are **equivalent** if  $\gamma_j = \gamma'_j$  for  $j \geq 3$  and  $\gamma_1 + \gamma_2 = \gamma'_1 + \gamma'_2$ . Let  $n = \nu_m(Q_3)$ ; note that  $\alpha_2 = n$ . Moreover  $[k_2 : k] | n$  and  $[k_2 : k] = n$  if and only if  $\beta_1 | \beta_2$ .

**Corollary 3.3.4** *Consider two standard monomials  $\prod_{j=1}^i Q_j^{\gamma_j}$  and  $\prod_{j=1}^i Q_j^{\gamma'_j}$  such that*

*$(\gamma_i, \gamma_{i-1}, \dots, \gamma_3) <_{lex} (\gamma'_i, \gamma'_{i-1}, \dots, \gamma'_3)$ . We have*

$$\nu_m \left( \prod_{j=3}^i Q_j^{\gamma_j} \right) \leq \nu_m \left( \prod_{j=3}^i Q_j^{\gamma'_j} \right) - n.$$

Proof : Let  $j \geq 3$  be the greatest integer such that  $\gamma_j < \gamma'_j$ . We have

$$\begin{aligned} \nu_m \left( \prod_{j=3}^i Q_j^{\gamma'_j} \right) - \nu_m \left( \prod_{j=3}^i Q_j^{\gamma_j} \right) &= \sum_{\ell=3}^j \gamma'_\ell \prod_{q=1}^{\ell-1} \alpha_q - \sum_{\ell=3}^j \gamma_\ell \prod_{q=1}^{\ell-1} \alpha_q \\ &= (\gamma'_j - \gamma_j) \prod_{q=1}^{\ell-1} \alpha_q + \sum_{\ell=3}^{j-1} (\gamma'_\ell - \gamma_\ell) \prod_{q=1}^{\ell-1} \alpha_q. \end{aligned}$$

**Claim :** For  $c_\ell < \alpha_\ell$ , we have

$$\sum_{\ell=3}^{j-1} c_\ell \prod_{q=1}^{\ell-1} \alpha_q < \prod_{q=1}^{j-1} \alpha_q. \quad (119)$$

Proof of Claim : By induction on  $j$ . For  $j = 4$ , the inequality is immediate. Assume the Claim is true for  $j - 1$ . The left hand side of (119) can be rewritten as

$$\sum_{\ell=3}^{j-1} c_\ell \prod_{q=1}^{\ell-1} \alpha_q = \sum_{\ell=3}^{j-2} c_\ell \prod_{q=1}^{\ell-1} \alpha_q + c_{j-1} \prod_{q=1}^{j-2} \alpha_q < \prod_{q=1}^{j-2} \alpha_q + c_{j-1} \prod_{q=1}^{j-2} \alpha_q \leq \prod_{q=1}^{j-1} \alpha_q.$$

The Claim is proved.

The monomials being standard,  $\gamma_\ell, \gamma'_\ell < \alpha_\ell$ , so  $\gamma'_\ell - \gamma_\ell > -\alpha_\ell$  and applying the Claim, we deduce that

$$\sum_{\ell=3}^{j-1} (\gamma'_\ell - \gamma_\ell) \prod_{q=1}^{\ell-1} \alpha_q > - \prod_{q=1}^{j-1} \alpha_q.$$

Since  $\gamma'_j - \gamma_j \geq 1$ , we get

$$(\gamma'_j - \gamma_j) \prod_{q=1}^{\ell-1} \alpha_q + \sum_{\ell=3}^{j-1} (\gamma'_\ell - \gamma_\ell) \prod_{q=1}^{\ell-1} \alpha_q > 0.$$

Each term being an integer divisible by  $\alpha_2$ , the former is greater or equal to  $\alpha_2 = n$ .  $\square$

Let  $(x, y)$  be a regular system of parameters of  $A$  such that  $\nu_\alpha(x) = \nu_\alpha(\mathbf{m})$  and  $\nu_\beta(x) = \nu_\beta(\mathbf{m})$ .

Recall from ([45], Appendix 5) that the **weak transform**  $I' \subset A'$  of an ideal  $I \subset A$  is defined by  $I' = x^{-a}IA'$  where  $a = \nu_{\mathbf{m}}(I)$ .

**Theorem 3.3.5** *Let  $\pi : A \rightarrow A'$  be a local blowing up with respect to  $\nu_\alpha$ . We assume that  $\pi$  is also a local blowing up with respect to  $\nu_\beta$ . Let  $\alpha'$  and  $\beta'$  be the strict transforms of  $\alpha$  and  $\beta$ . Then the separating ideal  $\langle \alpha', \beta' \rangle$  is equal to the weak transform of  $\langle \alpha, \beta \rangle$ .*

Proof : Since by hypothesis,  $\alpha', \beta'$  are both centered at a maximal ideal  $\mathbf{m}'$ , we have  $\langle \alpha, \beta \rangle \not\subseteq \mathbf{m}$ . In particular,  $x \notin \langle \alpha, \beta \rangle$ , hence  $x$  does not change sign between  $\alpha$  and  $\beta$ . Then  $f \in A$  changes sign between  $\alpha$  and  $\beta$  if and only if  $x^{-a}f$  changes sign between  $\alpha'$  and  $\beta'$ .

Since  $\langle \alpha, \beta \rangle$  is generated by elements changing sign between  $\alpha$  and  $\beta$ , it's weak transform is generated by elements which change sign between  $\alpha'$  and  $\beta'$ ; hence the weak transform of  $\langle \alpha, \beta \rangle$  is contained in  $\langle \alpha', \beta' \rangle$ .

To prove the opposite inclusion, let  $I' = \langle \alpha', \beta' \rangle$  and let  $I$  be the inverse transform of  $I'$ , that is the unique complete ideal of  $A$  whose weak transform is  $I'$  ([45], Appendix 5, p. 388). It remains to prove that  $I \subseteq \langle \alpha, \beta \rangle$ .

In order to do this, it suffices to find an element  $z \in I$  which changes sign between  $\alpha$  and  $\beta$  and such that  $\nu_\alpha(z) = \nu_\alpha(I)$ .

Let  $J_+$  be the greatest  $\nu_\alpha$ -ideal of  $A'$  whose  $\nu_\alpha$ -value is strictly greater than  $\nu_\alpha(I)$ . Note that  $\frac{IA'}{J_+ \cap IA'}$  is a  $k_{\nu_\alpha}$ -vector space. Let  $b_1, \dots, b_\ell, b_j = \prod_{r=1}^i Q_r^{\gamma_{jr}}$ , where  $i$  is the maximal index of the approximate roots  $Q_s$  involved, be a set of elements of  $I$  which induces a basis of  $\frac{IA'}{J_+ \cap IA'}$ , each monomial being standard. Moreover, since  $x$  divides  $y$  in  $A'$ , we may assume  $\gamma_{j2} = 0$  for all  $j$ . We may also suppose that  $b_1$  is the monomial for which the vector of exponents is maximal in the lexicographical order.

Let  $\tilde{z} \in I'$  be such that  $\nu_\alpha(\tilde{z}) = \nu_\alpha(I')$  and  $\tilde{z}$  changes sign between  $\alpha'$  and  $\beta'$ . Let  $z^\dagger = x^a \tilde{z}$ . Then  $z^\dagger \in IA'$  and  $\nu_\alpha(z^\dagger) = \nu_\alpha(IA') = \nu_\alpha(I)$ . Write  $z^\dagger = \sum_{j=1}^\ell z_j b_j$ . We may suppose  $z_1 = 1$ .

By the corollary (3.3.4), we have, for  $j \geq 2$ ,

$$\nu_{\mathfrak{m}}\left(\prod_{r=3}^i Q_j^{\gamma_{jr}}\right) \leq \nu_{\mathfrak{m}}\left(\prod_{r=3}^i Q_j^{\gamma_{j1}}\right) - n \leq a - n.$$

Write  $z^\dagger = b_1 + \sum_{j=2}^\ell (z_j x^n)(x^{-n} b_j)$ . Then, denoting by  $\bar{z}_j$  the image of  $z_j$  in the residual field  $k_{\nu_\alpha}$ , write  $\bar{z}_j = \sum_{t=0}^{n-1} c_t \left(\frac{y}{x}\right)^t$  where  $c_t \in k$ . So letting  $a_t$  be an element of  $A$  such that  $\bar{a}_t = c_t$  and  $v_j \in A$  be the element  $v_j = \sum_{t=0}^{n-1} a_t y^t x^{n-t}$ , we have

$$\nu_\alpha(v_j - z_j x^n) > n\nu_\alpha(x). \quad (120)$$

**Lemma 3.3.6** For  $j \geq 2$ ,  $x^{-n} b_j \in A$ .

Proof of lemma : By corollary (3.3.3),  $\nu_{\mathfrak{m}}(b_j) > \nu_{\mathfrak{m}}(b_1)$  and by corollary (3.3.4),  $\nu_{\mathfrak{m}}(\prod_{j=3}^i Q_j^{\gamma_{jr}}) \leq \nu_{\mathfrak{m}}(\prod_{j=3}^i Q_j^{\gamma_{j1}}) - n$ .

Now  $\gamma_{j1} = \nu_{\mathfrak{m}}(b_j) - \nu_{\mathfrak{m}}(\prod_{j=3}^i Q_j^{\gamma_{jr}}) > \nu_{\mathfrak{m}}(b_1) - \nu_{\mathfrak{m}}(\prod_{j=3}^i Q_j^{\gamma_{j1}}) + n \geq n$ .  $\square$

So  $z = b_1 + \sum_{j=2}^\ell v_j(x^{-n} b_j) \in IA' \cap A = I$  (because  $I$  is a contracted ideal). Since  $z^\dagger$  changes sign between  $\alpha'$  and  $\beta'$  and in view of (120),  $z$  changes sign between  $\alpha$  and  $\beta$ . This ends the proof of the theorem.  $\square$

**Remark 3.3.7** If  $B = R[x, y]$ . Let

$$B \rightarrow R[x_1, y_1] \rightarrow \cdots \rightarrow R[x_i, y_i]$$

be a sequence of blowings up induced by (115). Let  $C_i$  be the preimage of  $C$  (see (91)) in  $\text{Sper } R[x_i, y_i]$ . By proposition (3.2.17), there exist monomials  $\omega_1, \dots, \omega_s, \epsilon_1, \dots, \epsilon_s, \theta_1, \dots, \theta_t, \lambda_1, \dots, \lambda_t$  in  $x_i, y_i$  such that

$$C_i = \left\{ \delta \in \text{Sper} R[x_i, y_i] \left| \begin{array}{l} \nu_\delta(\omega_k) < \nu_\delta(\epsilon_k), \quad k \in \{1, \dots, s\} \\ \nu_\delta(\theta_j) = \nu_\delta(\lambda_j), \quad j \in \{1, \dots, t\} \\ \text{sgn}_\delta(x_i) = \text{sgn}_\alpha(x_i), \text{sgn}_\delta(y_i) = \text{sgn}_\alpha(y_i) \end{array} \right. \right\}.$$

By connectedness theorem ([21]),  $C_i$  is connected, hence so is  $C$ . This completes the proof of the Connectedness Conjecture for  $R[x, y]$  and so provides a new proof of the classical Pierce-Birkhoff Conjecture in dimension 2.

Let  $A$  be a regular 2-dimensional local ring with regular parameters  $(x, y)$ . Consider the set  $C'$  defined by the inequalities

$$\left| \sum_{j=1}^{m_i} b_{ji} \mathbf{Q}^{\theta_{ji}} \right| >_\delta n_i |\mathbf{Q}^{\epsilon_{j'i}}| \quad \forall i \in \{1, \dots, r\}, \quad \forall j' \in \{1, \dots, n_i\} \quad (121)$$

and the two sign conditions appearing in (91). Let  $C'_\ell$  be the preimage of  $C'$  in  $\text{Sper} A_\ell$ .

Rather than prove connectedness of  $C'_\ell$ , we will prove that  $\alpha^{(\ell)}$  and  $\beta^{(\ell)}$  lie in the same connected component of  $C'_\ell$ ; this will imply that  $\alpha$  and  $\beta$  lie in the same connected component of  $C'$ . Let  $\epsilon$  denote the common specialization of  $\alpha^{(\ell)}$  and  $\beta^{(\ell)}$ ; our assumptions imply that



$\mathfrak{p}_\epsilon = \mathfrak{m}_\ell$ . Let  $U$  be the subset of  $C'_\ell$  consisting of all the generalizations of  $\epsilon$  lying in  $C'_\ell$ . It is sufficient to prove that  $\alpha^{(\ell)}$  and  $\beta^{(\ell)}$  lie in the same connected component of  $U$ .

There are two cases to consider.

**Case 1.** Only one component of the exceptional divisor (that is the inverse image  $\rho_{\ell-1}^{-1}(\mathfrak{m})$ ) passes through  $\eta_\ell$ .

**Case 2.** Two components of the exceptional divisor pass through  $\eta_\ell$ .

Let  $(x_\ell, y_\ell)$  be a regular system of parameters of  $(A_\ell)_{\mathfrak{m}_\ell}$  such that the local equation of the exceptional divisor at  $\eta_\ell$  is  $x_\ell = 0$  in case 1 and  $x_\ell y_\ell = 0$  in case 2.

By Zariski's theory of complete ideals, for any  $f \in A \setminus \langle \alpha, \beta \rangle$ , the strict transform of  $f$  in  $A_\ell$  is a unit. In other words,  $f$  has the form  $f = x_\ell^n v$  in case 1 (resp.  $f = x_\ell^n y_\ell^m v$  in case 2) where  $v$  denotes a unit in  $(A_\ell)_{\mathfrak{m}_\ell}$ .

Hence the inequalities (121), appearing in the definition of  $C'$ , become inequalities of the form  $x_\ell^n > 0$  (resp.  $x_\ell^n y_\ell^m > 0$ ).

**Lemma 3.3.8** *Let  $E$  be an irreducible component of the exceptional divisor passing through  $\eta_\ell$ , defined by  $x_\ell = 0$ . There exists  $f \in A \setminus \langle \alpha, \beta \rangle$  such that  $f = x_\ell^n v$ ,  $v$  is a unit of  $(A_\ell)_{\mathfrak{m}_\ell}$  and  $n$  is odd.*

Proof : Let  $j \in \{1, \dots, \ell - 1\}$  be such that  $E$  is the strict transform in  $X_\ell$  of  $\pi_{j-}^{-1}(\eta_{j-1})$ . Let  $\nu_j$  be the divisorial valuation corresponding to  $E$ ; this valuation is defined as follows : for each  $f \in A_\ell$ , write  $f = x_\ell^n g$  such that  $x_\ell \nmid g$  in  $(A_\ell)_{\mathfrak{m}_\ell}$ , then  $\nu_j(f) = n$ .

Let  $\mathfrak{m} = \mathfrak{p}_0 \supset \dots \supset \mathfrak{p}_j$  the complete list of simple  $\nu_j$ -ideals given by Zariski's theory of complete ideals. Note that, since  $j < \ell$ ,  $\mathfrak{p}_j \supset \langle \alpha, \beta \rangle$ .

It follows from Zariski's factorization theorem for complete ideals that  $\nu_j(A \setminus \{0\})$  is generated by  $\nu_j(\mathfrak{p}_0), \dots, \nu_j(\mathfrak{p}_j)$ . Since the value group of  $\nu_j$  is  $\mathbb{Z}$ , the semigroup  $\nu_j(A \setminus \{0\})$  contains all the sufficiently large integer. Hence one of  $\nu_j(\mathfrak{p}_0), \dots, \nu_j(\mathfrak{p}_j)$  is odd.  $\square$

The lemma shows that  $x_\ell$  does not change sign between  $\alpha^{(\ell)}$  and  $\beta^{(\ell)}$  in Case 1 (resp. neither  $x_\ell$  nor  $y_\ell$  change sign between  $\alpha^{(\ell)}$  and  $\beta^{(\ell)}$  in Case 2).

Let  $\tilde{U} = \{\delta \in \text{Sper} A_\ell \mid \text{sgn}(x_\ell(\delta)) = \text{sgn}(x_\ell(\alpha)), \epsilon \in \overline{\{\delta\}}\}$  in Case 1 and  $\tilde{U} = \{\delta \in \text{Sper} A_\ell \mid \text{sgn}(x_\ell(\delta)) = \text{sgn}(x_\ell(\alpha)), \text{sgn}(y_\ell(\delta)) = \text{sgn}(y_\ell(\alpha)), \epsilon \in \overline{\{\delta\}}\}$  in Case 2. The above reasoning shows that  $\alpha^{(\ell)}, \beta^{(\ell)} \in \tilde{U} \subset U$ .

To prove the Definable Connectedness Conjecture (resp. the Connectedness Conjecture for excellent  $A$ ), it remains to prove the definable connectedness of  $\tilde{U}$  (resp. connectedness of  $\tilde{U}$  whenever  $A$  is excellent).

We are now ready to prove the two versions of the Connectedness Conjecture.

**Proposition 3.3.9** *Let  $A$  be an excellent regular local 2-dimensional ring. Let  $C$  be as in (91). Then  $C$  is connected.*

Proof : Let  $\epsilon, \ell$  and  $\tilde{U}$  as above. By the above considerations, it is sufficient to prove that  $\tilde{U}$  is connected. Thus it remains to prove the following lemma.

**Lemma 3.3.10** *Let  $A$  be an excellent regular  $n$ -dimensional local ring,  $x_1, \dots, x_n$  regular parameters of  $A$ . Fix a subset  $T \subset \{1, \dots, n\}$  and let  $D = \{\delta \in \text{Sper} A \mid x_i(\delta) > 0, i \in T \text{ and } \epsilon \in \overline{\{\delta\}}\}$ . Then  $D$  is connected.*

Proof :  $\epsilon$  determines an order on  $k$ . Let  $R$  denote the real closure of  $k$  relative to this order. Consider the natural homomorphisms

$$A \rightarrow \hat{A} = k[[X_1, \dots, X_n]] \xrightarrow{\sigma} R[[X_1, \dots, X_n]] \quad (122)$$

where  $\sigma$  is induced by  $\epsilon$ .

Let  $\hat{\epsilon}$  denote the point of  $\text{Sper } \hat{A}$  such that  $\mathfrak{p}_{\hat{\epsilon}} = (X_1, \dots, X_n)$  and  $\leq_{\hat{\epsilon}}$  is the total ordering of  $k$  given by  $\epsilon$ .

Following ([3], proposition 8.6),  $D$  is connected if and only if

$$\hat{D} = \{\delta \in \text{Sper } k[[X_1, \dots, X_n]] \mid X_i(\delta) > 0, i \in T, \hat{\epsilon} \in \overline{\{\delta\}}\}$$

is connected (this is where we are using the fact that  $A$  is excellent). Moreover,  $\hat{D}$  is homeomorphic to

$$\tilde{D} = \{\delta \in \text{Sper } R[[X_1, \dots, X_n]] \mid X_i(\delta) > 0, i \in T\}.$$

So it suffices to prove that  $\tilde{D}$  is connected.

By ([3], proposition 8.6),  $\tilde{D}$  is connected if and only if the set

$$D^\dagger = \{\delta \in \text{Sper } R[X_1, \dots, X_n]_{(X_1, \dots, X_n)} \mid X_i(\delta) > 0, i \in T, \delta \text{ is centered at } (X_1, \dots, X_n)\}$$

is connected.

Define

$$D_0 = \{\delta \in \text{Sper } R[X_1, \dots, X_n] \mid X_i(\delta) > 0, i \in T, \delta \text{ is centered at } (X_1, \dots, X_n)\}$$

and

$$D_{loc} = \{\delta \in \text{Sper } R[X_1, \dots, X_n]_{X_1 \dots X_n} \mid X_i(\delta) > 0, i \in T, \delta \text{ is centered at } (X_1, \dots, X_n)\}.$$

Now the maps  $\phi$  and  $\psi$  below induce homeomorphisms  $\phi|_{D_0} : D_0 \cong D_{loc}$  and  $\psi|_{D_0} : D_0 \cong D^\dagger$ .

$$\begin{array}{ccc} R[X_1, \dots, X_n] & \xrightarrow{\phi} & R[X_1, \dots, X_n]_{X_1 \dots X_n} \\ \psi \downarrow & & \\ R[X_1, \dots, X_n]_{(X_1, \dots, X_n)} & & \end{array}$$

So it suffices to prove that  $D_{loc}$  is connected. But

$$D_{loc} = \bigcap_{N \in \mathbb{N}} D_N$$

where

$$D_N = \{\delta \in \text{Sper } R[X_1, \dots, X_n]_{X_1 \dots X_n} \mid \frac{1}{N} \geq X_i(\delta) \geq 0, i \in T\}.$$

By Proposition 7.5.1. of [7], each  $D_N$  is a non empty closed connected subset of  $\text{Sper } R[X_1, \dots, X_n]_{X_1 \dots X_n}$ , hence  $D_{loc}$  is connected by ([21], lemma 7.1).  $\square$

This completes the proof of the Connectedness and Pierce-Birkhoff Conjecture for any excellent 2-dimensional ring  $A$ .

**Remark 3.3.11** *The above proof is a special case of the following general principle. Let  $A$  be an excellent regular local ring with regular parameters  $x = (x_1, \dots, x_n)$  and real closed residue field  $R$ . We have natural morphisms*

$$\begin{array}{ccc} \text{Sper } A & \xleftarrow{\phi} & \text{Sper } R[[X_1, \dots, X_n]] \\ & & \downarrow \pi \\ & & \text{Sper } R[X_1, \dots, X_n]_{(X_1, \dots, X_n)} \end{array}$$

*Let  $D \subset \text{Sper } A$  be a constructible set such that all the elements of  $A$  appearing in the definition of  $D$  belong to  $A \cap R[X_1, \dots, X_n]_{(X_1, \dots, X_n)}$ . Let  $\hat{D} = \phi^{-1}(D)$ , let  $U$  be the subset of all points of  $\text{Sper } R[X_1, \dots, X_n]_{(X_1, \dots, X_n)}$  centered at the origin. Let  $D_{\text{pol}}$  be the subset of  $U$  defined by the same formulas as  $D$ . By ([3], proposition 8.6), to show that  $D$  is connected, it is enough to prove that  $D_{\text{pol}}$  is connected.*

*In many cases, this principle applies also to nested intersection  $D = \bigcap_{N \in \mathbb{N}} D_N$  of constructible sets defined by elements of  $A \cap R[X_1, \dots, X_n]_{(X_1, \dots, X_n)}$ .*

*This allows to generalize all the results of ([21]) from the case of polynomial rings to that of arbitrary excellent regular local rings with real closed residue fields.*

### 3.4 Proof of the Definable Connectedness Conjecture

Next we prove the Definable Connectedness Conjecture, hence the Pierce-Birkhoff Conjecture, without the excellence hypothesis on  $A$ .

**Proposition 3.4.1** *Let  $(A, \mathfrak{m}, k)$  be a regular local 2-dimensional ring,  $(x, y)$  a regular system of parameters. The sets*

$$U = \{\delta \in \text{Sper } A \mid x(\delta) > 0, \epsilon \in \overline{\{\delta\}}\} \tag{123}$$

$$V = \{\delta \in \text{Sper } A \mid x(\delta) > 0, y(\delta) > 0, \epsilon \in \overline{\{\delta\}}\} \tag{124}$$

*are definably connected.*

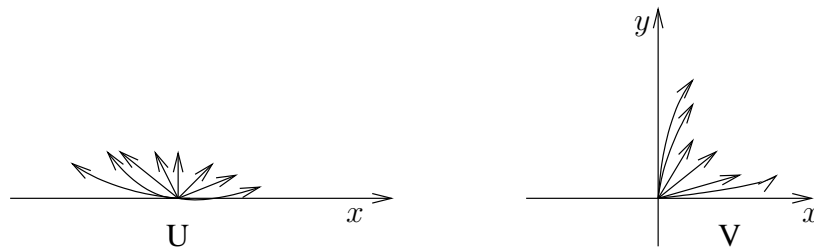


FIGURE 1 – The sets  $U$  and  $V$

Proof : By contradiction. Write  $U = F \amalg G$ ,  $F = \bigcup F_i$ ,  $G = \bigcup G_i$  where  $F_i, G_i$  are basic open sets. Each  $F_i$  and  $G_i$  is defined by finitely many inequalities of the form  $g > 0$ ,  $g \in A$ . Let  $g_1, \dots, g_r \in A$  be the list of elements of  $A$ , appearing in the definition of all of  $F_i$  and  $G_i$ , and  $x$ . The proof of the proposition will be given after a few auxiliary definitions and results.

**Definition 3.4.2** Let  $\text{Sper } A \leftarrow X_1 \leftarrow \cdots \leftarrow X_t$  be a sequence of point blowings up. Let  $X_t = \bigcup_{j=1}^s \text{Sper } A_{jt}$  be the open covering of  $X_t$ , given by the definition of real geometric surface. We say that  $g_1, \dots, g_r$  are simultaneously locally monomial in  $X_t$  if for all  $j \in \{1, \dots, s\}$  and any maximal ideal  $\mathfrak{m}' \subset A_{jt}$ , there exists a regular system of parameters  $(x', y')$  of  $A' := (A_{jt})_{\mathfrak{m}'}$  such that all of  $g_1, \dots, g_r$  are monomials in  $(x', y')$  multiplied by units of  $(A_{jt})_{\mathfrak{m}'}$ .

Let  $g_1, \dots, g_r \in A$  be as in the proof of Proposition 3.4.1 and let  $\text{Sper } A \leftarrow X_1 \leftarrow \cdots \leftarrow X_t$  be a sequence of point blowings up such that  $g_1, \dots, g_r$  are simultaneously locally monomial in  $X_t$ , denote by  $\rho_j : X_t \rightarrow \text{Sper } A$  the composition of all the morphisms in that sequence (with the notations following (114)). Let  $U^{(t)} = \rho_t^{-1}(U)$ ,  $F^{(t)} = \rho_t^{-1}(F)$ ,  $G^{(t)} = \rho_t^{-1}(G)$ .

Take a point  $\delta \in \rho_t^{-1}(\epsilon)$ , let  $A', \mathfrak{m}', x', y', A_{jt}$  be as in the definition of simultaneously locally monomial.

**Definition 3.4.3** We say that  $\delta$  is a special point of  $\rho_t^{-1}(\epsilon)$  if  $ht(\mathfrak{p}_\delta) = 2$  and  $\{x'y' = 0\} = \rho_t^{-1}(\epsilon) \cup \{g_1 \cdots g_r = 0\}$  locally near  $\delta$ .

Given a special point  $\delta \in \rho_t^{-1}(\epsilon)$  and  $(u', v')$  a regular system of parameters at  $\delta$ , let

$$C(\delta, u', v') = \{\gamma \in X_t \mid u'(\gamma) > 0, v'(\gamma) > 0, \delta \in \overline{\{\gamma\}}\}.$$

**Lemma 3.4.4** Take a point  $\xi \in \rho_t^{-1}(\epsilon)$ , not lying on the strict transform of  $\{x = 0\}$ . Let  $A', \mathfrak{m}', x', y', A_{jt}$  be as in the definition of simultaneously locally monomial.

Assume that the set  $\rho_t^{-1}(\epsilon)$  is defined in  $U = {}^{(t)}U = \rho_t^{-1}(U)$  locally near  $\xi$  by  $x' = 0$ . Let

$$E = \{\delta \in \rho_t^{-1}(\epsilon) \mid k \subset A_{jt}(\delta) \text{ is an inclusion of ordered fields}\}.$$

Let  $\delta_1$  be the greatest special point in  $E$  such that  $\delta_1 < \xi$ . If no special point of  $E$  is strictly smaller than  $\xi$ , we write by convention  $\delta_1 = -\infty$ .

Let  $\delta_2$  be the smallest special point of  $E$  such that  $\delta_2 > \xi$ . If no special element of  $E$  is strictly greater than  $\xi$ , we write by convention  $\delta_2 = +\infty$ .

For  $i \in \{1, 2\}$ , let  $(x', v'_i)$  be a regular system of parameters at  $\delta_i$  such that  $\{v'_i > 0\} \cap (\delta_1, \delta_2) \neq \emptyset$ .

Then the set

$$D(\delta_1, \delta_2) = C(\delta_1, x', v'_1) \cup C(\delta_2, x', v'_2) \cup \{\delta \in U^{(t)} \mid x'(\delta) > 0, \overline{\{\delta\}} \cap (\delta_1, \delta_2) \neq \emptyset\}$$

is contained either in  $F^{(t)}$  or in  $G^{(t)}$ .

Proof : Assume  $\xi$  is not special. Let  $F_{\dagger} = \overline{F^{(t)}} \cap [\delta_1, \delta_2]$  and  $G_{\dagger} = \overline{G^{(t)}} \cap [\delta_1, \delta_2]$ .  $F_{\dagger}, G_{\dagger}$  are relatively closed in  $[\delta_1, \delta_2]$  and  $[\delta_1, \delta_2]$  is connected (lemma (3.2.8)), so  $F_{\dagger} \cap G_{\dagger} \neq \emptyset$ .

Take a point  $\eta \in F_{\dagger} \cap G_{\dagger}$ . Replacing  $\eta$  by its specialization, we may assume that  $\eta$  is closed. For each  $i \in \{1, \dots, r\}$ , locally near  $\eta$ , write

$$g_i = x'^a g'_i, x' \nmid g'_i \text{ if } \eta \notin \{\delta_1, \delta_2\} \text{ and}$$

$$g_i = x'^a y'^b g'_i, x' \nmid g'_i, y' \nmid g'_i \text{ if } \eta = \delta_\ell, \ell \in \{1, 2\} \text{ where } y' = v'_i.$$

Take an open set  $W$ , containing  $\eta$ , such that for all  $\delta \in W$  and all  $i \in \{1, \dots, r\}$ , we have

$$\text{sgn}(g'_i(\delta)) = \text{sgn}(g'_i(\eta)). \tag{125}$$

Since  $\eta \in \overline{F^{(t)}} \cap \overline{G^{(t)}}$ , there exist  $\delta \in F^{(t)} \cap W$ ,  $\gamma \in G^{(t)} \cap W$  and an  $i \in \{1, \dots, r\}$  such that  $g_i$  changes sign between  $\delta$  and  $\gamma$ . Since  $x'$  (reps.  $x', y'$ ) does not change sign between  $\gamma$  and  $\delta$  this contradicts 125.

Assume  $\xi$  is special and  $(x', y')$  is a regular local system of parameters as in the definition. Let  $\delta \in D(\delta_1, \delta_2)$  be the unique point such that  $x'(\delta) > 0, y'(\delta) = 0$ . We have  $\{\delta\} = D(\delta_1, \delta_2) \cap \{y' = 0\}$ . Then

$$D(\delta_1, \delta_2) = \{\delta\} \coprod D(\delta_1, \xi) \coprod D(\xi, \delta_2).$$

Let  $\delta_- \in D(\delta_1, \xi)$  be the unique point such that  $x'(\delta_-) > 0, y'(\delta_-) < 0$  and  $|y'(\delta_-)|^N < |x'(\delta_-)|, \forall N \in \mathbb{N}$ . Then  $\delta \in \overline{\{\delta_-\}}$ , in particular,

$$\delta \in \overline{D(\delta_1, \xi)}. \quad (126)$$

Similarly

$$\delta \in \overline{D(\xi, \delta_2)}. \quad (127)$$

By the previous case, each of  $D(\delta_1, \xi), D(\xi, \delta_2)$  is contained either in  $F^{(t)}$  or  $G^{(t)}$ .

Without loss of generality, assume that  $D(\delta_1, \xi) \subset F^{(t)}$ . By (126) and the relative closedness of  $F^{(t)}$ , we have  $\delta \in F^{(t)}$ . By (127) and the relative closedness of  $G^{(t)}$ , we have  $D(\xi, \delta_2) \subset F^{(t)}$ , so  $D(\delta_1, \delta_2) \subset F^{(t)}$  as desired.  $\square$

**Corollary 3.4.5** *Let  $[\delta_1, \delta_2]$  be a maximal interval, then  $D([\delta_1, \delta_2])$  is entirely contained either in  $F^{(t)}$  or in  $G^{(t)}$ .*

Proof : This follows from the preceding lemma by induction on the number of special points inside  $[\delta_1, \delta_2]$ .

In order to address the global connectedness, we need a notion of **signed dual graph** associated to a sequence of point blowings up of a point  $\epsilon \in \text{Sper } A$  and a subset  $W$  of  $\text{Sper } A$ .

For each maximal interval  $I$ , take  $\text{Sper } A_{ji} \subset X$  such that  $I \setminus \text{Sper } A_{ji}$  is either empty or consists of one distinguished point. Pick and fix  $x'_I$  such that  $I \cap \text{Sper } A_{ji} \subset \{x'_I = 0\}$ .

Consider two pairs  $(I, s), (\tilde{I}, \tilde{s})$  where  $I$  is a  $j$ -maximal interval,  $\tilde{I}$  a  $\tilde{j}$ -maximal interval and  $s, \tilde{s} \in \{+, -\}$ . We say that these two pairs are equivalent if the following conditions hold :

- (a)  $I \cap \text{Sper } A_{ji} \cap \text{Sper } A_{\tilde{j}i} = \tilde{I} \cap \text{Sper } A_{ji} \cap \text{Sper } A_{\tilde{j}i}$ ,
- (b) the sets

$$\{\delta \in \text{Sper } A_{ji} \cap \text{Sper } A_{\tilde{j}i} \mid \text{sgn}(x_I(\delta)) = s\}$$

and

$$\{\delta \in \text{Sper } A_{ji} \cap \text{Sper } A_{\tilde{j}i} \mid \text{sgn}(x_{\tilde{I}}(\delta)) = \tilde{s}\}$$

coincide in a neighbourhood of  $I \cap \text{Sper } A_{ji} \cap \text{Sper } A_{\tilde{j}i}$ ,

- (c)  $W^{(i)} \cap \text{Sper } A_{ji} \subset \{\delta \in \text{Sper } A_{ji} \mid \text{sgn}(x'_I(\delta)) = s, \overline{\{\delta\}} \cap I \neq \emptyset\} \neq \emptyset$ ,
- (d)  $W^{(i)} \cap \text{Sper } A_{\tilde{j}i} \subset \{\delta \in \text{Sper } A_{\tilde{j}i} \mid \text{sgn}(x'_{\tilde{I}}(\delta)) = \tilde{s}, \overline{\{\delta\}} \cap \tilde{I} \neq \emptyset\} \neq \emptyset$ .

**Definition 3.4.6** 1. *A vertex of the signed dual graph  $\Gamma_i$  associated to  $X_i$  and  $W$  is an equivalence class of a pair  $(I, s)$ , which we will still denote, by abuse of notation,  $(I, s)$ .*

2. *By definition, two distinct vertices  $(I, s)$  and  $(\tilde{I}, \tilde{s})$  of  $\Gamma_i$  are connected by an edge of  $\Gamma_i$  if the following conditions hold :*

(a)  *$I$  and  $\tilde{I}$  share a common endpoint  $\xi$  and  $(x_I, x_{\tilde{I}})$  form a regular system of parameters at  $\xi$ ;*

(b) *write  $I = [\xi, \delta_I], \tilde{I} = [\xi, \delta_{\tilde{I}}]$ , then*

$$W^{(i)} \cap \{\delta \in X_i \mid \text{sgn}(x'_I(\delta)) = s, \text{sgn}(x'_{\tilde{I}}(\delta)) = \tilde{s}, \overline{\{\delta\}} \ni \xi\} \neq \emptyset.$$

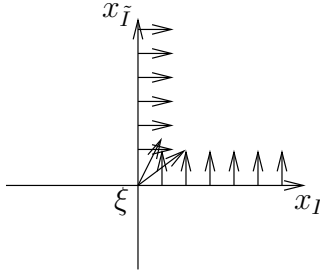


FIGURE 2 – This figure represents an edge of  $\Gamma_i$  connecting two vertices  $(I, s)$  and  $(\tilde{I}, \tilde{s})$ . Here  $I = [0, \infty]$ ,  $\tilde{I} = [0, \infty]$ ,  $s = \tilde{s} = +$ .

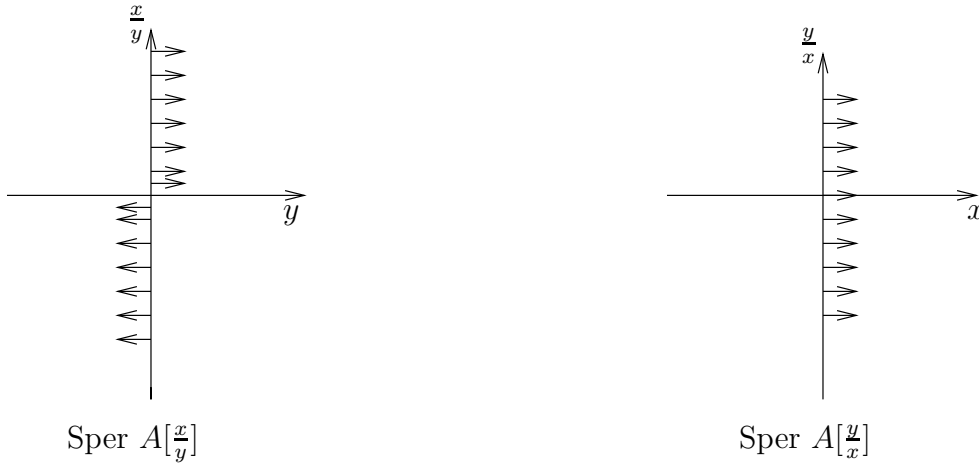


FIGURE 3 – This figure shows the set  $U^{(1)}$  in the affine charts

Example :  $\Gamma_1$  consists of one vertex and no edge (see Fig. (3)).

The following proposition describes the transformation law from  $\Gamma_i$  to  $\Gamma_{i+1}$  in the case when  $W = U$  or  $W = V$ .

**Proposition 3.4.7** *Consider the point blowing up  $\pi_i : X_{i+1} \rightarrow X_i$ . Let  $\xi$  be the center of the blowing up.*

(1) *Let  $a = (I, s)$ ,  $b = (\tilde{I}, \tilde{s})$  two vertices of  $\Gamma_i$  connected by an edge  $(a, b)$  and suppose  $\xi$  is the point common to  $I$  and  $\tilde{I}$ . Then  $\Gamma_{i+1}$  is obtained from  $\Gamma_i$  by performing the following operation for each pair of vertices  $(a, b)$  as above : remove the edge  $(a, b)$  and add a new vertex  $c$  and two new edges  $(a, c)$  and  $(b, c)$ .*

(2) *Suppose we are not in case (1). Let  $a = (I, s)$  where  $I = [\delta_1, \delta_2]$ .*

- *Case 2.1 :  $\xi \notin \{\delta_1, \delta_2\}$  :*

(a) *If  $a$  belongs to two edges  $(a, b)$ ,  $(a, c)$  of  $\Gamma_i$ , remove  $a$  and the two edges  $(a, b)$ ,  $(a, c)$ . Introduce three new vertices  $d, e, f$  and four new edges  $(b, d)$ ,  $(d, e)$ ,  $(e, f)$ ,  $(f, c)$ .*

(b) *If  $a$  belongs to only one edge  $(a, b)$ , remove  $a$  and the edge  $(a, b)$ . Introduce three new vertices  $d, e, f$  and three new edges  $(b, d)$ ,  $(d, e)$ ,  $(e, f)$ .*

(c) *If  $a$  belongs to only one edge (in other words, if  $i = 1$ ) then  $\Gamma_2 = \bullet \text{---} \bullet \text{---} \bullet$  is a chain of three vertices and two edges*

- Case 2.2 :  $\xi \in \{\delta_1, \delta_2\}$  :

(a)  $i = 1$  and  $W = U$ , then  $\Gamma_2 = \bullet \text{---} \bullet \text{---} \bullet$  is a chain of three vertices and two edges

(b)  $i > 1$  or  $W = V$ , then each vertex  $a = (I, s)$  such that  $\xi \in I$  is an endpoint of  $\Gamma_i$ .

For each such vertex  $a$ , we add a new vertex  $b$  and a new edge  $(a, b)$ .

Proof : (1) Let  $\delta'_I = \pi_i^{-1}(\delta_I)$ ,  $\delta'_{\tilde{I}} = \pi_i^{-1}(\delta_{\tilde{I}})$ ,  $\{x'_I = 0\}$  the strict transform of  $\{x_I = 0\}$ ,  $\{x'_{\tilde{I}} = 0\}$  the strict transform of  $\{x_{\tilde{I}} = 0\}$ ,  $\xi_a = \{x'_I = 0\} \cap \pi_i^{-1}(\xi)$ ,  $\xi_b = \{x'_{\tilde{I}} = 0\} \cap \pi_i^{-1}(\xi)$ .

Let  $a_{i+1}, b_{i+1}, c_{i+1}$  be the vertices of  $\Gamma_{i+1}$  defined by  $a_{i+1} = ([\xi_a, \delta'_I], s)$ ,  $b_{i+1} = ([\xi_b, \delta'_{\tilde{I}}], \tilde{s})$ ,  $c_{i+1} = (J, \sigma)$  where  $J = [\xi_a, \xi_b]$ . Let  $x_J$  be a defining equation of the exceptional divisor and  $\sigma$  is the sign of  $x_J$  on the set  $\{sgn(x_I) = s, sgn(x_{\tilde{I}}) = \tilde{s}\}$  near  $\xi$ .

The edges joining those vertices are straightforward.

(2) Case 2.1 : (a) Let  $(x_I, y')$  be a regular system of parameters at  $\xi$ . Assume that

$$y'(\delta_1) < 0, \quad (128)$$

$$y'(\delta_2) > 0. \quad (129)$$

Let  $\delta'_i = \pi^{-1}(\delta_i)$ ,  $i \in \{1, 2\}$ . Let  $\xi' = \pi^{-1}(\xi) \cap \{x'_I = 0\}$  and  $d = ([\delta'_1, \xi'], -s)$ ,  $f = ([\delta'_2, \xi'], s)$ ,  $e = ([-\infty, +\infty], s)$  where  $[-\infty, +\infty] = \{x_I = 0\}$  in  $\text{Sper}(A_{ji}[\frac{y'}{x_I}])$ . Again, the edges are clearly defined. This proves 2.1(a). The cases 2.1.(b) and (c) is similar but easier.

Case 2.2 : (a) Let  $a = (I, s)$  and  $(x_I, y')$  be a regular system of parameters at  $\xi$ . Put  $y' = \frac{x}{y}$ ,  $x_I = y$ . Let  $A_{11} = A[x_I, y']$ . The point  $\xi \in \text{Sper } A_{11}$  is the unique point such that  $\text{supp}(\xi) = (x_I, y')$  and which induces the given order on  $k$ . Let  $A_{12} = A_{11}[x'_I, y']$  where  $x'_I = \frac{x_I}{y'}$ .

Let  $I' \subset \{x'_I = 0\}$  be the 1-maximal interval given by  $-\infty \leq y' \leq +\infty$  and  $\tilde{I}' \subset \{y' = 0\}$  be the 1-maximal interval given by  $0 \leq x'_I \leq +\infty$ .

Now the vertices of  $\Gamma_2$  are  $(\tilde{I}', +)$ ,  $(I', +)$ ,  $(\tilde{I}', -)$  with the edges clearly defined.

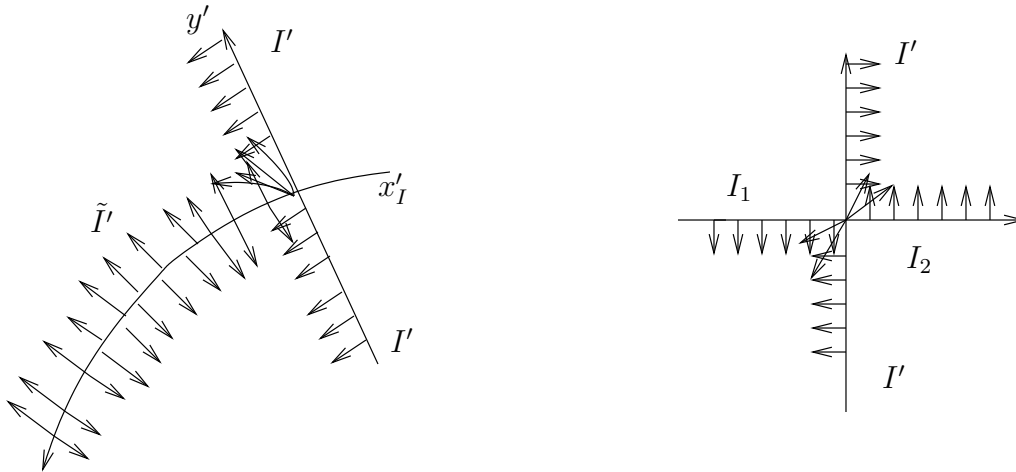


FIGURE 4 – This figure shows the set  $U^{(2)}$  in the cases 2.2.a and 2.1.c respectively

(b) Let  $a = (I, s)$  be a vertex such that  $\xi \in I$  is an endpoint of  $\Gamma_i$ . Suppose that  $\xi \in A_{ji}$ . Let  $(x_I, y')$  be a regular system of parameters at  $\xi$ . Let  $A_{j+1} = A_{ji}[x'_I, y']$  where  $x'_I = \frac{x_I}{y'}$ . Without loss of generality, assume that  $x_I > 0, y' > 0$  on  $W^{(i)}$ .

Let  $I' \subset \{x'_I = 0\}$  be the strict transform of  $I$  in  $\text{Sper } A_{j+1}$ , it is an  $(i + 1)$ -maximal interval. Let  $\tilde{I}' \subset \{y' = 0\}$  be the  $(i + 1)$ -maximal interval given by  $0 \leq x'_I \leq +\infty$ . Now the



new vertex  $b$  added to  $\Gamma_{i+1}$  is  $(\tilde{I}', +)$ . It is connected by an edge to  $a$  which is represented in  $\text{Sper } A_{j_{i+}}$  by  $(I', +)$ .  $\square$

**Corollary 3.4.8**  $\Gamma_i$  is connected.

Proof : By induction on  $i$ . For  $i = 1$ , the graph consisting of one vertex is connected. The induction step follows from the proposition.  $\square$

Let us finish the proof of proposition (3.4.1). To each vertex  $(I = [\delta_1, \delta_2], s)$  of  $\Gamma_i$  we associate the set  $D(\delta_1, \delta_2) \subset U^{(i)}$  which by corollary (3.4.5) is entirely contained in  $F^{(i)}$  or  $G^{(i)}$ . This defines a partition  $\Gamma_F = \{(I, s) \mid D(\delta_1, \delta_2) \subset F^{(i)}\}$ ,  $\Gamma_G = \{(I, s) \mid D(\delta_1, \delta_2) \subset G^{(i)}\}$  of the set of vertices of  $\Gamma_i$ . Assume that  $\Gamma_F \neq \emptyset$  and  $\Gamma_G \neq \emptyset$ . Since  $\Gamma_i$  is connected, there exist  $a = ([\delta_{1a}, \delta_{2a}], s_a) \in \Gamma_F$ ,  $b = ([\delta_{1b}, \delta_{2b}], s_b) \in \Gamma_G$  such that  $(a, b)$  is an edge of  $\Gamma_i$ . Then  $D(\delta_{1a}, \delta_{2a}) \subset F^{(i)}$ ,  $D(\delta_{1b}, \delta_{2b}) \subset G^{(i)}$  and  $D(\delta_{1a}, \delta_{2a}) \cap D(\delta_{1b}, \delta_{2b}) \neq \emptyset$ . This is a contradiction. This concludes the proof of proposition (3.4.1).  $\square$

## Références

- [1] S. Abhyankar and T.T. Moh, *Newton–Puiseux expansion and generalized Tschirnhausen transformation I*. *Reine Angew. Math.* 260, 47–83 (1973).
- [2] S. Abhyankar and T.T. Moh, *Newton–Puiseux expansion and generalized Tschirnhausen transformation II*. *Reine Angew. Math.* 261, 29–54 (1973).
- [3] C. Andradas, L. Bröcker, J.M. Ruiz, *Constructible Sets in Real Geometry*, Springer (1996).
- [4] D. Alvis, B. Johnston, J.J. Madden, *Local structure of the real spectrum of a surface, infinitely near points and separating ideals*. Preprint.
- [5] R. Baer, *Über nicht-archimedisch geordnete Körper* (Beiträge zur Algebra). *Sitz. Ber. Der Heidelberger Akademie*, 8 Abhandl. (1927).
- [6] G. Birkhoff and R. Pierce, *Lattice-ordered rings*. *Annales Acad. Brasil Ciênc.* 28, 41–69 (1956).
- [7] J. Bochnak, M. Coste, M.-F. Roy, *Géométrie algébrique réelle*. Springer–Verlag, Berlin 1987.
- [8] S.D. Cutkosky, B. Teissier, *Semi-groups of valuations on local rings*, *Michigan Mat. J.*, Vol. 57, pp. 173–193 (2008).
- [9] C. N. Delzell, *On the Pierce–Birkhoff conjecture over ordered fields*, *Rocky Mountain J. Math.*
- [10] L. Fuchs, *Telweise geordnete algebraische Strukturen*. Vandenhoeck and Ruprecht, 1966.
- [11] Herrera Govantes, F. J. ; Olalla Acosta, M. A. ; Spivakovsky, M. *Valuations in algebraic field extensions*, *Journal of Algebra* , Vol. 312 , N. 2 , pp. 1033–1074 (2007).
- [12] R. Goldin and B. Teissier, *Resolving singularities of plane analytic branches with one toric morphism*.
- [13] F. J. Herrera Govantes, M. A. Olalla Acosta, M. Spivakovsky, B. Teissier, *Extending a valuation centered in a local domain to the formal completion*. Preprint.
- [14] M. Henriksen and J. Isbell, *Lattice-ordered rings and function rings*. *Pacific J. Math.* 11, 533–566 (1962).
- [15] I. Kaplansky, *Maximal fields with valuations I*. *Duke Math. J.*, 9 :303–321 (1942).
- [16] I. Kaplansky, *Maximal fields with valuations II*. *Duke Math. J.*, 12 :243–248 (1945).

- [17] W. Krull, Allgemeine Bewertungstheorie, J. Reine Angew. Math. 167, 160–196 (1932).
- [18] T.C. Kuo, *Generalized Newton–Puiseux theory and Hensel’s lemma in  $\mathbb{C}[[x, y]]$* . Canadian J. Math., (6) XLI, 1101–1116 (1989).
- [19] T.C. Kuo, *A simple algorithm for deciding primes in  $\mathbb{C}[[x, y]]$* . Canadian J. Math., 47 (4), 801–816 (1995).
- [20] M. Lejeune-Jalabert, *Thèse d’Etat*. Université Paris 7 (1973).
- [21] F. Lucas, J.J. Madden, D. Schaub and M. Spivakovsky, *On connectedness of sets in the real spectra of polynomial rings*, Manuscripta Math. 128, 505–547, 2009.
- [22] F. Lucas, D. Schaub and M. Spivakovsky, *On the Pierce–Birkhoff conjecture in dimension 3*, in preparation.
- [23] S. MacLane, *A construction for prime ideals as absolute values of an algebraic field*. Duke Math. J. 2, 492–510 (1936).
- [24] S. MacLane, *A construction for absolute values in polynomial rings*. Transactions of the AMS 40, 363–395 (1936).
- [25] S. MacLane and O.F.G. Schilling, *Zero-dimensional branches of rank one on algebraic varieties*. Ann. of Math. 40, 3 (1939).
- [26] J. J. Madden, *Pierce–Birkhoff rings*. Arch. Math. 53, 565–570 (1989).
- [27] J. J. Madden, preprint.
- [28] L. Mahé, *On the Pierce–Birkhoff conjecture*. Rocky Mountain J. Math. 14, 983–985 (1984).
- [29] M. Marshall, *Orderings and real places of commutative rings*. J. Alg. 140, 484–501 (1991).
- [30] M. Marshall *The Pierce–Birkhoff conjecture for curves*. Can. J. Math. 44, 1262–1271 (1992).
- [31] H. Matsumura *Commutative Algebra*. Benjamin/Cummings Publishing Co., Reading, Mass., 1970.
- [32] A. Prestel *Lectures on formally real fields*, Lecture Notes in Math., Springer–Verlag—Berlin, Heidelberg, New York, 1984.
- [33] A.Prestel, C.N.Delzell *Positive Polynomials*, Springer monographs in mathematics, 2001.
- [34] S. Priess-Crampe, *Angeordnete Strukturen : gruppen, körper, projektive Ebenen*, Springer–Verlag—Berlin, Heidelberg, New York, 1983.
- [35] N. Schwartz, *Real closed spaces*. Habilitationsschrift, München 1984.
- [36] M. Spivakovsky, *Valuations in function fields of surfaces*. Amer. J. Math 112, 1, 107–156 (1990).
- [37] M. Spivakovsky, *A solution to Hironaka’s polyhedra game*. Arithmetic and Geometry, Vol II, Papers dedicated to I. R. Shafarevich on the occasion of his sixtieth birthday, M. Artin and J. Tate, editors, Birkhäuser, 1983, pp. 419–432.
- [38] B. Teissier, *Valuations, deformations and toric geometry.*, Proceedings of the Saskatoon Conference and Workshop on valuation theory, Vol. II, F-V. Kuhlmann, S. Kuhlmann, M. Marshall, editors, Fields Institute Communications, 33 (2003), 361–459.
- [39] M. Vaquié, *Famille admise associée à une valuation de  $K[x]$* , Séminaires et Congrès 10, edited by Jean-Paul Brasselet and Tatsuo Suwa, 2005, pp. 391–428.
- [40] M. Vaquié, *Extension d’une valuation*, Trans. Amer. Math. Soc. 359 (2007), no. 7, 3439–3481.

- [41] M. Vaquié, *Algèbre graduée associée à une valuation de  $K[x]$* , Adv. Stud. Pure Math., 46, Math. Soc. Japan, Tokyo, 2007.
- [42] M. Vaquié, *Famille admissible de valuations et défaut d'une extension*, J. Algebra 311 (2007), no. 2, 859–876.
- [43] M. Vaquié, *Valuations*, 539–590, Progr. Math., 181, Birkhäuser, Basel, 2000.
- [44] S. Wagner, *On the Pierce-Birkhoff Conjecture for Smooth Affine Surfaces over Real Closed Fields*, ArXiv :0810.4800, 2009.
- [45] O. Zariski, P. Samuel *Commutative Algebra*, Vol. II, Springer Verlag