Restricted ramification in dihedral and fake Z_p-extensions

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RESTRICTED RAMIFICATION IN DIHEDRAL AND FAKE $\mathbb{Z}_p$-EXTENSIONS

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Abstract. We study restricted ramification in dihedral extensions of number fields, with a focus on its ℓ-adic properties along fake $\mathbb{Z}_p$-extensions, for p and ℓ distinct odd primes. Our results generalise previous works in the (pro)-dihedral setting, with applications to the study of more general fake Iwasawa theories beyond the classical case of standard class groups.

1. Introduction

Dihedral extensions of number fields have long been a key object of interest, not only because they represent the "first step" beyond abelian extensions, whose arithmetic is controlled by class field theory, but also in light of the role that they play in the theory of complex multiplication: for a very explicit example of the connection between class field theory for imaginary quadratic fields and dihedral Galois representations arising from CM modular forms, a reader might consult [BN18, Théorème 1.1]. It is therefore natural to extend to the dihedral setting the analysis of "restricted ramification" that is normally performed in the abelian case through class field theory.

The aim of Section 2 is to establish a formula relating the orders of the T-class groups in a dihedral extension, generalising the main result of [CN20] (see Theorem 2.12 below). Although some arguments can be borrowed from *ibid.*, one needs to take into accounts that class field theory for the regular class group and for the T-class group $Cl^T_\bullet$ are different, because the local cohomology of the units at the ramified places that lie in T is different in the two cases. Beyond analysing these local contributions in detail, we also devise a general strategy different from the one in [CN20] that allows us to replace global cohomology arguments with local ones altogether. Beside its intrinsic interest, this generalisation is motivated by applications to Iwasawa theory. Indeed, studying Iwasawa-theoretic properties in geometric settings (as for elliptic curves or, more generally, modular forms), typically requires to focus on Galois groups with restricted ramification: this is discussed in detail in by Coates–Sujatha in [CS10], and specific examples where this is needed are in the definition of the fine Selmer groups in [NS23, Theorem 4.20] or in [NOR23, Theorem 4.3].

In Section 3, we consider the setting of a fake $\mathbb{Z}_p$-extension, as first defined in [CN23] (see Definition 3.1 below for the precise definition). Prototypical examples of fake $\mathbb{Z}_p$-extensions are provided by anticyclotomic $\mathbb{Z}_p$-extensions: given an imaginary quadratic field F, the anticyclotomic $\mathbb{Z}_p$-extensions L_∞/F is the unique $\mathbb{Z}_p$-extension of F that is Galois but non abelian over $\mathbb{Q}$, and all its layers L_n are normal extensions of the rationals with $\mathrm{Gal}(L_n/\mathbb{Q}) \cong D_{p^n}$. It gives rise to a tower $\{K_n/\mathbb{Q}\}_{n\geq 0}$ of extensions of degree p^n that are not Galois, being the fixed fields of the non normal subgroup of order 2 in D_{p^n}: this tower is a fake $\mathbb{Z}_p$-extension. While it is a classical result that many p-adic arithmetic properties of F are encoded in the behaviour of class groups along the anticyclotomic $\mathbb{Z}_p$-extension (we refer the reader to the fundamental work [Rub88] and references therein), recent advances show the importance of analysing primes above from p, as in [KNS24, Theorem 3.1], or in [KL23, Theorem A] (see also the introduction of [BL23] for a thorough literature review). The aim of this section is to generalise some results obtained in [CN23] about fake $\mathbb{Z}_p$-extensions of dihedral type, even beyond the anticyclotomic cases, but for primes ℓ different from p.

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We conclude this introduction by observing that a full formalization of the results in this paper in the proof assistant Lean[1] through its mathematical library Mathlib (see [mC20]) should be within reach. The definition of the ideal class group, together with its finiteness for number fields, has recently been obtained (see [BDNN21] and [BDNN22] for a longer version); the full formalization of the topological group of the idèles and of its class group is discussed in [dFF22]; in the recent work [dFFN24] de Frutós-Fernandez and the author implemented the basics of the arithmetic of local fields; finally, Livingston has developed in [Liv23] the foundations of group cohomology in Lean, thus providing the full technical machinery to implement results about dihedral extensions and their class field theory in Lean.

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I take this opportunity to correct two typos introduced by the copy-editor while preparing the publisher version of my joint work [CN20]:

p. 5, line -5: $\dots c_\delta$ is multiplied by -1 on $H^2(G,\mathbb{Z})\dots \to \dots c_\delta$ is multiplication by -1 on $H^2(G,\mathbb{Z})\dots$
p. 11, line 1: $\dots$since $\mathrm{N}_{L/K}\circ\iota^T_{L/K}$ is multiplied by $2=[L:K]$ $\dots \to \dots$since $\mathrm{N}_{L/K}\circ\iota^T_{L/K}$ is multiplication by $2=[L:K]$ $\dots$

2. A formula for T-class groups in dihedral extensions

The main result of [CN20] is a formula expressing the ratio of the class numbers in dihedral extensions of number fields:

Theorem 2.1 ([CN20, Theorem 3.14])**.** *Let L/k be a Galois extension of number fields whose Galois group $D=\mathrm{Gal}(L/k)$ is dihedral of order $2q$ with q odd. Let $\Sigma\subseteq D$ be a subgroup of order 2 and let $G\subseteq D$ be the subgroup of order q. Set $K=L^\Sigma$ and $F=L^G$. Then*

$$\frac{|Cl_L|\cdot|Cl_k|^2}{|Cl_F|\cdot|Cl_K|^2}=\frac{\widehat{h}^0\big(D,\mathcal{O}_L^\times[\frac12]\big)}{\widehat{h}^{-1}\big(D,\mathcal{O}_L^\times[\frac12]\big)}=\frac{\widehat{h}^0(D,\mathcal{O}_L^\times)}{\widehat{h}^{-1}(D,\mathcal{O}_L^\times)}\frac{\widehat{h}^{-1}(\Sigma,\mathcal{O}_L^\times)}{\widehat{h}^0(\Sigma,\mathcal{O}_L^\times)},$$

where, for any group Γ and any Γ-module B, $\widehat{h}^i(\Gamma,B)$ denotes the order of the Tate cohomology group $\widehat{H}^i(\Gamma,B)$, and $B[\frac12]$ denotes $B\otimes\mathbb{Z}[\frac12]$.

Given any number field[2] M and any finite set T of primes of M, we introduce the following notation

Notation 2.2. $\mathbb{A}_M^\times$ is the group of idèles of M, and $\mathcal{C}_M=\mathbb{A}_M^\times/M^\times$ is its idèle class group. The group of prime-to-T fractional ideals — isomorphic to the free abelian group generated by the primes not in T — is denoted by Id_M^T and Pr_M^T is its subgroup of prime-to-T principal fractional ideals. The T-class group of M is the quotient $\mathrm{Id}_M^T/\mathrm{Pr}_M^T$ and is denoted Cl_M. The group of T-units E_M^T is defined as

$$E_M^T=\{x\in M^\times \text{ such that } v_\mathfrak{p}(x)=0 \text{ for all finite primes } \mathfrak{p}\notin T\},$$

while the symbol $\mathcal{U}_M^T$ denotes the group of idèles of valuation 0 at every finite place not in T,

$$\mathcal{U}_M^T=\prod_{\mathfrak{p}\in T}M_\mathfrak{p}^\times\times\prod_{\mathfrak{p}\notin T}\mathcal{O}_{M_\mathfrak{p}}^\times\times\prod_{v|\infty}M_v^\times.$$

[1] see https://lean-lang.org/.
[2] In this paper, by "number field" we mean a *finite* field extension of $\mathbb{Q}$.

The S-class group of M appears in the bottom-right corner of the following commutative diagram, analogous to [CN20, Diagram (3.6)],

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & E_M^T & \longrightarrow & M^\times & \longrightarrow & \mathrm{Pr}_M^T & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathcal{U}_M^T & \longrightarrow & \mathbb{A}_M^\times & \longrightarrow & \mathrm{Id}_M^T & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & Q_M^T & \longrightarrow & \mathcal{C}_M & \longrightarrow & Cl_M^T & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array} \tag{2.1}$$

where Q_M^T is defined by the exactness of the diagram. It is easy to see that Cl_M^T is the quotient of Cl_M modulo the subgroup generated by the classes of the primes in T. Moreover, given a finite Galois extension of number fields M'/M, all groups appearing in (2.1) are $\mathrm{Gal}(M'/M)$-modules, and the diagram is exact and commutative in the category of $\mathrm{Gal}(M'/M)$-modules. We refer the reader to [Tat67] for the main results of class field theory that we use.

The following is an adaptation of the notation introduced in [CN20, §3]:

Notation 2.3. For an extension M'/M of number fields, $\iota_{M_2/M_1}^T : Cl_{M_1}^T \to Cl_{M_2}^T$ and $\mathrm{N}_{M_2/M_1} : Cl_{M_2}^T \to Cl_{M_1}^T$ denote the map induced by extending ideals from M_1 to M_2 and by taking the norm of ideals from M_2 to M_1, respectively.

Given a finite group Γ, we denote by Nm_Γ the element

$$\mathrm{Nm}_\Gamma = \sum_{g\in\Gamma} g \ \in \mathbb{Z}[\Gamma];$$

for a Γ-module B, we let the same symbol denote the map $\mathrm{Nm}_\Gamma \colon B \to B$ induced by multiplication by Nm_Γ. Similarly, we let $B[\mathrm{Nm}_\Gamma]$ denote the kernel of multiplication by Nm_Γ. Writing $\widehat{H}^i(\Gamma, B)$ (*resp.* $H^i(\Gamma, B)$) for the Tate (*resp.* ordinary) i-th cohomology group of Γ with values in B, the symbol $\widehat{h}^i(\Gamma, B)$ (*resp.* $h^i(\Gamma, B)$) denotes the order of $\widehat{H}^i(\Gamma, B)$ (*resp.* of $H^i(\Gamma, B)$). We occasionally write B^Γ to denote the group $H^0(\Gamma, B)$.

One of our first results is a generalisation of [Nuc10, Proposition 2.2] to the setting of T-class groups.

Lemma 2.4. *Let M'/M be a Galois extension of number fields, with Galois group $\Gamma = \mathrm{Gal}(M'/M)$. Then there are isomorphisms*

$$\mathrm{Ker}(\iota_{M'/M}^T) \cong \mathrm{Ker}\big(H^1(\Gamma, E_{M'}^T) \to H^1(\Gamma, \mathcal{U}_{M'}^T)\big) \qquad \text{and} \qquad \mathrm{Coker}(\iota_{M'/M}^T) \cong H^1(\Gamma, Q_{M'}^T).$$

Proof. Taking Γ-cohomology of the bottom row of (2.1) yields the exact sequence

$$0 \longrightarrow \big(Q_{M'}^T\big)^\Gamma \longrightarrow \big(\mathcal{C}_{M'}\big)^\Gamma \longrightarrow \big(Cl_{M'}^T\big)^\Gamma \longrightarrow H^1(\Gamma, Q_{M'}^T) \longrightarrow H^1(\Gamma, \mathcal{C}_{M'}) = 0. \tag{2.2}$$

The sequence (2.2) sits in the second line of the following exact, commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Q_M^T & \longrightarrow & \mathcal{C}_M & \longrightarrow & Cl_M^T & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow{\scriptstyle \iota_{M'/M}^T} & & \\
0 & \longrightarrow & \big(Q_{M'}^T\big)^\Gamma & \longrightarrow & \big(\mathcal{C}_{M'}\big)^\Gamma & \longrightarrow & \big(Cl_{M'}^T\big)^\Gamma & \longrightarrow & H^1(\Gamma, Q_{M'}^T) \longrightarrow 0
\end{array}$$

and the snake lemma yields isomorphisms

$$\mathrm{Ker}(\iota^T_{M'/M}) \cong \big(Q^T_{M'}\big)^\Gamma/Q^T_M \qquad \text{and} \qquad \mathrm{Coker}(\iota^T_{M'/M}) \cong H^1(\Gamma, Q^T_{M'}) \tag{2.3}$$

already establishing the second isomorphism in the statement. Taking Γ-cohomology of the left-most column in (2.1) gives rise to the commutative diagram of exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E^T_M & \longrightarrow & \mathcal{U}^T_M & \longrightarrow & Q^T_M & \longrightarrow & 0 \\ & & \| & & \| & & \downarrow & & \\ 0 & \longrightarrow & \big(E^T_{M'}\big)^\Gamma & \longrightarrow & \big(\mathcal{U}^T_{M'}\big)^\Gamma & \longrightarrow & \big(Q^T_{M'}\big)^\Gamma & \longrightarrow & H^1(\Gamma, E^T_{M'}) \longrightarrow H^1(\Gamma, \mathcal{U}^T_{M'}) \end{array}$$

By applying the snake lemma again, we get an isomorphism

$$\big(Q^T_{M'}\big)^\Gamma/Q^T_M \cong \mathrm{Ker}\big(H^1(\Gamma, E^T_{M'}) \to H^1(\Gamma, \mathcal{U}^T_{M'})\big)$$

which, combined with the first isomorphism in (2.3), yields the first isomorphism of the statement. □

From now on in this section, we work in the dihedral setting and we need the following notation:

Notation 2.5. Fix be a number field k and let L/k be a Galois extension whose Galois group $D = \mathrm{Gal}(L/k)$ is isomorphic to D_q the dihedral group of order $2q$, where q is any *odd* number. Write $G \lhd D$ for the normal subgroup of order q and fix a subgroup $\Sigma < D$ of order 2. Fix also the choice of a generator ρ of G and σ of Σ, letting Σ' denote the subgroup $\langle \sigma\rho \rangle < D$, again of order 2. Let F/k be the subextension of L/k fixed by G and let K/k (*resp.* K'/k) be the subextension of L/k fixed by Σ (*resp.* fixed by Σ'). Finally, we let T be a fixed, finite set of primes in k and we denote by $T_s \subseteq T$ its subset of primes that split in F/k.

The first ingredient for the proof of Theorem 2.12 is a generalisation of [CN20, Proposition 3.2], extending the result in *loc. cit.* to T-class groups. Before stating it, observe that there are equalities

$$\big(Cl^T_L[\tfrac{1}{2}]\big)^\Sigma = \iota^T_{L/K}\big(Cl^T_K[\tfrac{1}{2}]\big) \qquad \text{and} \qquad \big(Cl^T_L[\tfrac{1}{2}]\big)^{\Sigma'} = \iota^T_{L/K'}\big(Cl^T_{K'}[\tfrac{1}{2}]\big) : \tag{2.4}$$

to see this (we provide the argument for K, the one for K' being analogous), observe first that $\iota^T_{L/K}$ is injective when restricted to $Cl^T_K[\frac{1}{2}]$, because its composition with $\mathrm{N}_{L/K}\colon Cl^T_L[\frac{1}{2}] \to Cl^T_K[\frac{1}{2}]$ coincides with the multiplication by 2. Moreover, the "algebraic" norm $\mathrm{Nm}_\Sigma\colon Cl^T_L \to Cl^T_L$ can be factored as

$$\mathrm{Nm}_\Sigma\colon Cl^T_L \xrightarrow{\mathrm{N}_{L/K}} Cl^T_K \xrightarrow{\iota^T_{L/K}} Cl^T_L.$$

Hence, given any class $c \in \big(Cl^T_L[\frac{1}{2}]\big)^\Sigma$, we have $2c = c + \sigma c = \mathrm{Nm}_\Sigma c = \iota^T_{L/K}(\mathrm{N}_{L/K}c)$: since multiplication by 2 is an automorphism, this implies that $\iota^T_{L/K}$ is also surjective, establishing (2.4).

Proposition 2.6. *In the setting of Notation 2.5, there is an equality*

$$\frac{|Cl^T_L[\frac{1}{2}]|}{|Cl^T_K[\frac{1}{2}]|^2} = \frac{h^0(G, Cl^T_L[\frac{1}{2}])}{h^0(D, Cl^T_L[\frac{1}{2}]) \cdot \widehat{h}^{-1}(D, Cl^T_L[\frac{1}{2}]) \cdot |\mathrm{Nm}_D Cl^T_L[\frac{1}{2}]|}.$$

Proof. Let $\eta^T\colon Cl^T_K \oplus Cl^T_{K'} \to Cl^T_L$ be the map defined by

$$\eta^T(c, c') = \iota^T_{L/K}(c)\iota^T_{L/K'}(c') \qquad \text{for} \qquad c \in Cl^T_K, c' \in Cl^T_{K'}.$$

The proposition will follow from the following two equalities:

$$\big(Cl^T_L[\tfrac{1}{2}] : \mathrm{Im}(\eta^T)[\tfrac{1}{2}]\big) = \frac{h^0(Cl^T_L[\frac{1}{2}],)}{\widehat{h}^{-1}(D, Cl^T_L[\frac{1}{2}]) \cdot |\mathrm{Nm}_D Cl^T_L[\frac{1}{2}]|} \tag{2.5}$$

and

$$\big|\mathrm{Ker}\,\eta^T[\tfrac{1}{2}]\big| = h^0(D, Cl^T_L[\tfrac{1}{2}]). \tag{2.6}$$

Indeed, comparing orders in the tautological exact sequence

$$0 \longrightarrow \operatorname{Ker}\eta^T[\tfrac{1}{2}] \longrightarrow Cl_K^T[\tfrac{1}{2}] \oplus Cl_{K'}^T[\tfrac{1}{2}] \longrightarrow Cl_L^T[\tfrac{1}{2}] \longrightarrow Cl_L^T[\tfrac{1}{2}]/\operatorname{Im}\eta^T[\tfrac{1}{2}] \longrightarrow 0$$

and using (2.5) and (2.6) yields the statement.

We start with the proof of (2.5). Before inverting 2, we can interpret the index in the left-hand side as

$$\big(Cl_L^T : \operatorname{Im}\eta^T\big) = \big(\operatorname{Nm}_G Cl_L^T : \operatorname{Nm}_G\big(\operatorname{Im}\eta^T\big)\big) \cdot \big(Cl_L^T[\operatorname{Nm}_G] : \operatorname{Im}\eta^T \cap Cl_L^T[\operatorname{Nm}_G]\big) \tag{2.7}$$

by applying equation [CN20, (3.4)] with $f = \operatorname{Nm}_G$, $B = Cl_L^T$ and $B' = \operatorname{Im}\eta^T$. Moreover, combining (2.4) with the equality $\operatorname{Im}\eta^T = \iota_{L/K}^T(Cl_K^T)\iota_{L/K}^T(Cl_{K'}^T)$, that follows from the definition of η^T, yields

$$\operatorname{Im}(\eta^T[\tfrac{1}{2}]) = \big(Cl_L^T[\tfrac{1}{2}]\big)^{\Sigma}\big(Cl_L^T[\tfrac{1}{2}]\big)^{\Sigma'}. \tag{2.8}$$

Replacing (2.8) in the right-most term of (2.7) (after inverting 2) we obtain

$$\big(Cl_L^T[\tfrac{1}{2}] : \operatorname{Im}(\eta^T[\tfrac{1}{2}])\big) = \big(\operatorname{Nm}_G Cl_L^T[\tfrac{1}{2}] : \operatorname{Nm}_G\big(\operatorname{Im}\eta^T[\tfrac{1}{2}]\big)\big) \cdot \Big(Cl_L^T[\tfrac{1}{2}][\operatorname{Nm}_G] : \big((Cl_L^T[\tfrac{1}{2}])^{\Sigma}(Cl_L^T[\tfrac{1}{2}])^{\Sigma'}\big) \cap Cl_L^T[\operatorname{Nm}_G]\Big).$$

The second isomorphism of [CN20, Lemma 2.5] (with $B = Cl_L^T[\frac{1}{2}]$) allows to simplify the right-hand side of the above equation to

$$\big(Cl_L^T[\tfrac{1}{2}] : \operatorname{Im}(\eta^T[\tfrac{1}{2}])\big) = \big(\operatorname{Nm}_G Cl_L^T[\tfrac{1}{2}] : \operatorname{Nm}_G\big(\operatorname{Im}\eta^T[\tfrac{1}{2}]\big)\big) \cdot h^1(D, Cl_L^T[\tfrac{1}{2}]). \tag{2.9}$$

Replacing again (2.8), this time in the *first* factor of the right-hand side of (2.9), gives the equality

$$\big(\operatorname{Nm}_G Cl_L^T[\tfrac{1}{2}] : \operatorname{Nm}_G\big(\operatorname{Im}\eta^T[\tfrac{1}{2}]\big)\big) = \Big(\operatorname{Nm}_G Cl_L^T[\tfrac{1}{2}] : \operatorname{Nm}_G\big((Cl_L^T[\tfrac{1}{2}])^{\Sigma}(Cl_L^T[\tfrac{1}{2}])^{\Sigma'}\big)\Big)$$

and since $\operatorname{Nm}_G\big(Cl_L^T[\frac{1}{2}]\big)^{\Sigma} = \operatorname{Nm}_G\big(Cl_L^T[\frac{1}{2}]\big)^{\Sigma'}$, this further simplifies to

$$\big(\operatorname{Nm}_G Cl_L^T[\tfrac{1}{2}] : \operatorname{Nm}_G\big(\operatorname{Im}\eta^T[\tfrac{1}{2}]\big)\big) = \Big(\operatorname{Nm}_G Cl_L^T[\tfrac{1}{2}] : \operatorname{Nm}_G\big(Cl_L^T[\tfrac{1}{2}]\big)^{\Sigma}\Big).$$

Applying again the equality $\operatorname{Nm}_\Sigma Cl_L^T[\frac{1}{2}] = \big(Cl_L^T[\frac{1}{2}]\big)^{\Sigma}$, we can continue the substitutions to obtain

$$\big(\operatorname{Nm}_G Cl_L^T[\tfrac{1}{2}] : \operatorname{Nm}_G\big(\operatorname{Im}\eta^T[\tfrac{1}{2}]\big)\big) = \big(\operatorname{Nm}_G Cl_L^T[\tfrac{1}{2}] : \operatorname{Nm}_G\operatorname{Nm}_\Sigma\big(Cl_L^T[\tfrac{1}{2}]\big)\big) = \big(\operatorname{Nm}_G Cl_L^T[\tfrac{1}{2}] : \operatorname{Nm}_D\big(Cl_L^T[\tfrac{1}{2}]\big)\big)$$

and inserting this back into (2.9) gives

$$\big(Cl_L^T[\tfrac{1}{2}] : \operatorname{Im}(\eta^T[\tfrac{1}{2}])\big) = \big(\operatorname{Nm}_G Cl_L^T[\tfrac{1}{2}] : \operatorname{Nm}_D\big(Cl_L^T[\tfrac{1}{2}]\big)\big) \cdot h^1(D, Cl_L^T[\tfrac{1}{2}]) = \frac{\big|\operatorname{Nm}_G Cl_L^T[\tfrac{1}{2}]\big| \cdot h^1(D, Cl_L^T[\tfrac{1}{2}])}{\big|\operatorname{Nm}_D\big(Cl_L^T[\tfrac{1}{2}]\big)\big|}. \tag{2.10}$$

By [CN20, Proposition 2.1] there is an equality

$$h^1(D, Cl_L^T[\tfrac{1}{2}]) = \frac{h^1(G, Cl_L^T[\tfrac{1}{2}])}{\widehat{h}^{-1}(D, Cl_L^T[\tfrac{1}{2}])}$$

whereas, by definition of Tate cohomology and thanks to the vanishing of the Herbrand quotient of a finite module,

$$\big|\operatorname{Nm}_G Cl_L^T[\tfrac{1}{2}]\big| = \frac{Cl_L^T[\tfrac{1}{2}]^G}{\widehat{h}^0(G, Cl_L^T[\tfrac{1}{2}])} = \frac{Cl_L^T[\tfrac{1}{2}]^G}{h^1(G, Cl_L^T[\tfrac{1}{2}])} :$$

thus we can rewrite the numerator in the right-most term of (2.10) as

$$\big|\operatorname{Nm}_G Cl_L^T[\tfrac{1}{2}]\big| \cdot h^1(D, Cl_L^T[\tfrac{1}{2}]) = \frac{h^0(G, Cl_L^T[\tfrac{1}{2}])}{h^1(G, Cl_L^T[\tfrac{1}{2}])} \cdot \frac{h^1(G, Cl_L^T[\tfrac{1}{2}])}{\widehat{h}^{-1}(D, Cl_L^T[\tfrac{1}{2}])} = \frac{h^0(G, Cl_L^T[\tfrac{1}{2}])}{\widehat{h}^{-1}(D, Cl_L^T[\tfrac{1}{2}])}.$$

Inserting the above equality in (2.10) finally yields (2.5).

Passing now to (2.6), we are actually going to show that the arrow

$$\theta\colon \operatorname{Ker}\eta^T[\tfrac{1}{2}] \longrightarrow H^0(D, Cl_L^T[\tfrac{1}{2}])$$

defined by $\theta(c,c') = \iota^T_{L/K}(c)$ is an isomorphism, as in the end of the proof of [CN20, Proposition 3.2]. That θ is well-defined follows from the equality $\iota^T_{L/K}c = \left(\iota^T_{L/K'}c'\right)^{-1} \in Cl^T_L$ (by definition of η^T), and therefore both Σ and Σ' fix $\iota^T_{L/K}(c)$, so this class is actually invariant under the whole D. That θ is injective has already been observed during the proof of (2.4), when we established that the restriction of $\iota^T_{L/K}$ to the prime-to-2 component is injective. To show that it is surjective, let $b \in H^0(D, Cl^T_L)$: in particular, b is fixed by both Σ and Σ' and we can write

$$b = \iota^T_{L/K}(c) = \iota^T_{L/K}(c')$$

for some $c \in Cl^T_K[\frac{1}{2}], c' \in Cl^T_{K'}[\frac{1}{2}]$, thanks to (2.8). It follows that the element (c,c') is in $\operatorname{Ker}\eta^T[\frac{1}{2}]$ and $b = \theta(c,c')$, concluding the proof that θ is an isomorphism and establishing (2.6). □

Before stating the next result, we introduce one more piece of notation:

Notation 2.7. Given any abelian group B, write $B^{(2)}$ to denote its 2-Sylow subgroup.

Remark 2.8. The restriction of the morphism $\iota^T_{L/F}$ to $\left(Cl^T_F\right)^{(2)}$ is injective: indeed, the composition $\mathrm{N}_{L/F} \circ \iota^T_{L/F}$ coincides with raising to the q-th power, that is an automorphism of the 2-Sylow subgroup. The same argument shows that the restrictions of $\iota^T_{K/k}$ and $\iota^T_{K'/k}$ to $\left(Cl^T_k\right)^{(2)}$ are injective: we thus identify the 2-Sylow subgroup $\left(\iota^T_{L/F}Cl^T_F\right)^{(2)}$ (*resp.* $\left(\iota^T_{K/k}Cl^T_k\right)^{(2)}$, *resp.* $\left(\iota^T_{K'/k}Cl^T_k\right)^{(2)}$) with $\left(Cl^T_F\right)^{(2)}$ (*resp.* with $\left(Cl^T_k\right)^{(2)}$).

Lemma 2.9. *The 2-adic valuations of the T-class groups in the dihedral extension L/k satisfy the relation*

$$v_2\left(\frac{|Cl^T_L|}{|Cl^T_F|}\right) = 2 \cdot v_2\left(\frac{|Cl^T_K|}{|Cl^T_k|}\right).$$

Proof. It follows from Walter's isomorphism (see [Wal77, Lemma 5.2]) that

$$\left|\left(Cl^T_L\right)^{(2)}/H^0(G, Cl^T_L)^{(2)}\right| = \left|H^0(\Sigma, Cl^T_L)^{(2)}/H^0(D, Cl^T_L)^{(2)}\right| \cdot \left|H^0(\Sigma', Cl^T_L)^{(2)}/H^0(D, Cl^T_L)^{(2)}\right|. \tag{2.11}$$

We claim that there are isomorphisms

$$H^0(\Sigma, Cl^T_L)^{(2)}/H^0(D, Cl^T_L)^{(2)} \cong \left(Cl^T_K/Cl^T_k\right)^{(2)}, \quad H^0(\Sigma', Cl^T_L)^{(2)}/H^0(D, Cl^T_L)^{(2)} \cong \left(Cl^T_{K'}/Cl^T_k\right)^{(2)}, \tag{2.12}$$

as well as

$$H^0(G, Cl^T_L)^{(2)} \cong \left(Cl^T_F\right)^{(2)}. \tag{2.13}$$

Granting (2.12) and (2.13), the lemma follows from (2.11).

Concerning (2.13), it is enough by Remark 2.8 to show that every class in $\left(Cl^T_L\right)^{(2)}$ fixed by G is in the image of $\iota^T_{L/F}$: identifying $H^0(G, Cl^T_L)^{(2)}$ with $H^0(G, \left(Cl^T_L\right)^{(2)})$ and then with $\mathrm{Nm}_G\left(Cl^T_L\right)^{(2)}$ (because $\widehat{H}^0(G, \left(Cl^T_L\right)^{(2)}) \cong H^2(G, \left(Cl^T_L\right)^{(2)}) = 0$), it is enough to show that every class of the form $\mathrm{Nm}_G x$ is in the image of $\iota^T_{L/F}$. This follows from the factorization $\mathrm{Nm}_G = \iota^T_{L/F} \circ \mathrm{N}_{L/F}$.

To prove (2.12) (we prove the relation for K, the proof for K' being analogous), consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \left(Cl^T_k\right)^{(2)} & \longrightarrow & \left(Cl^T_K\right)^{(2)} & \longrightarrow & \left(Cl^T_K/Cl^T_k\right)^{(2)} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \iota^T_{L/k}} & & \downarrow{\scriptstyle \iota^T_{L/K}} & & \downarrow & & \\
0 & \longrightarrow & H^0(D, Cl^T_L)^{(2)} & \longrightarrow & H^0(\Sigma, Cl^T_L)^{(2)} & \longrightarrow & H^0(\Sigma, Cl^T_L)^{(2)}/H^0(D, Cl^T_L)^{(2)} & \longrightarrow & 0
\end{array}$$

The isomorphisms of Lemma 2.4 combined with [CN20, Lemma 2.9] show that the kernels (*resp.* the cokernels) of the first and the second vertical arrow are isomorphic. The snake lemma yields the result. □

As remarked before Proposition 2.6, and as the proof of Lemma 2.4 somewhat exemplifies, the cohomological arguments in [CN20] generalise easily to the case of T-class groups, since they rely on the study of the commutative square (2.1), analogous to [CN20, (3.6)]. In particular, the proof of the two equalities in the following lemma is analogous to the corresponding ones *ibid.* and we simply sketch it.

Lemma 2.10. *Let $H \subseteq D$ be any subgroup and let $M = L^H$ be the subfield fixed by H. The equalities*

$$\left|Cl_M^T[\tfrac{1}{2}]\right| = \frac{h^0(H, Cl_L^T[\tfrac{1}{2}]) \cdot h^1(H, E_L^T[\tfrac{1}{2}])}{h^1(H, \mathcal{U}_L^T[\tfrac{1}{2}]) \cdot h^1(H, \mathrm{Pr}_L^T[\tfrac{1}{2}])}$$

and

$$|Cl_k^T[\tfrac{1}{2}]| = \frac{\widehat{h}^{-1}(D, Cl_L^T[\tfrac{1}{2}]) \cdot \left|\mathrm{Nm}_D Cl_L^T[\tfrac{1}{2}]\right| \cdot \widehat{h}^0(D, E_L^T[\tfrac{1}{2}])}{\widehat{h}^0(D, \mathcal{U}_L^T[\tfrac{1}{2}]) \cdot \widehat{h}^{-1}(D, \mathrm{Pr}_L^T[\tfrac{1}{2}])}$$

hold.

Proof. The proof of the first equality is the same as the proof of [CN20, Proposition 3.5], replacing Diagram (3.6) *ibid.* with (2.1). Similarly, the second equality is obtained through the proof of [CN20, Proposition 3.7], invoking our Lemma 2.4 instead of equation (3.14) *ibid.* □

Unlike Lemma 2.10, the following result requires some adaptation to take into account the primes in the set T. Recall that we introduced in Notation 2.5 the symbol T_s to denote the primes in T that split in F/k.

Proposition 2.11. *There is an equality*

$$h^1(G, \mathcal{U}_L^T[\tfrac{1}{2}]) = h^1(D, \mathcal{U}_L^T[\tfrac{1}{2}]) \cdot \widehat{h}^0(D, \mathcal{U}_L^T[\tfrac{1}{2}]) \cdot \prod_{\mathfrak{p} \in T_s} d_{\mathfrak{p}}$$

where $d_{\mathfrak{p}}$ is the local degree of a prime $\mathfrak{p}$ in the extension L/k.

Proof. Just for this proof, write $\mathcal{T}$ to denote the set of primes ideal in $\mathcal{O}_F$ dividing the primes in T. Accordingly, write $\mathrm{Ram}(L/F)$ (*resp.* $\mathrm{Ram}(L/k)$) for the set of primes in F (*resp.* in k) that ramify in L/F (*resp.* in L/k), and let $\mathrm{Ram}(L/F) \setminus \mathcal{T}$ (*resp.* $\mathrm{Ram}(L/k) \setminus T$) denote the set difference $\mathrm{Ram}(L/F) \setminus \big(\mathcal{T} \cap \mathrm{Ram}(L/F)\big)$ (*resp.* $\mathrm{Ram}(L/k) \setminus \big(T \cap \mathrm{Ram}(L/k)\big)$).

Given any prime ideal $\mathfrak{p}$ in k, choose and fix a prime ideal $\mathfrak{P}$ in L diving it, and write $G(\mathfrak{P})$ (*resp.* $D(\mathfrak{P})$) to denote the decomposition group of $\mathfrak{P}$ in the extension L/F (*resp.* L/k). Whenever $\mathfrak{p}$ splits in F/k we also fix a conjugate ideal $\widetilde{\mathfrak{P}}$ in L, dividing $\mathfrak{p}$ but not dividing $\mathfrak{P} \cap F$. Accordingly, given any prime $\mathfrak{l}$ in F, denote by $\mathfrak{L}$ the prime in L lying above $\mathfrak{l} \cap k$ that has been fixed with the previous choice. Finally, denote by $\mathcal{O}_{\mathfrak{P}}^{\times}$ — instead of $\mathcal{O}_{L_{\mathfrak{P}}}^{\times}$ — the local units at a prime $\mathfrak{P}$.

We begin with the left-hand side in the statement. The description in [Tat67, Proposition 7.2] of the Galois structure of the idèles yields an identification

$$H^1(G, \mathcal{U}_L^T) = \prod_{\mathfrak{l} \in \mathcal{T}} H^1(G(\mathfrak{L}), L_{\mathfrak{L}}^{\times}) \times \prod_{\mathfrak{l} \notin \mathcal{T}} H^1(G(\mathfrak{L}), \mathcal{O}_{\mathfrak{L}}^{\times}) \times \prod_{v|\infty} H^1(G(v), L_v^{\times}) = \prod_{\mathfrak{l} \notin \mathcal{T}} H^1(G(\mathfrak{L}), \mathcal{O}_{\mathfrak{L}}^{\times})$$

where the second equality follows from Hilbert 90. Since the cohomology of local units is trivial in unramified extension, the product above simplifies as

$$H^1(G, \mathcal{U}_L^T) = \prod_{\mathfrak{l} \in \mathrm{Ram}(L/F) \setminus \mathcal{T}} H^1(G(\mathfrak{L}), \mathcal{O}_{\mathfrak{L}}^{\times}) \cong \prod_{\mathfrak{l} \in \mathrm{Ram}(L/F) \setminus \mathcal{T}} \mathbb{Z}/e_{\mathfrak{l}}$$

where $e_{\mathfrak{l}}$ is the ramification degree of $\mathfrak{l}$ in the extension L/F, and the isomorphism follows from local class field theory. Given a prime $\mathfrak{l}$ occurring in the above product, let $\mathfrak{p} = \mathfrak{l} \cap k$; it lies in $\mathrm{Ram}(L/k) \setminus T$ and its ramification index $e_{\mathfrak{p}}$ in L/k satisfies

$$e_{\mathfrak{p}} = \begin{cases} e_{\mathfrak{l}} & \text{if } \mathfrak{p} \text{ splits or is inert in } F/k; \\ 2e_{\mathfrak{l}} & \text{if } \mathfrak{p} \text{ is ramified in } F/k. \end{cases}$$

In particular,

$$h^1(G,\mathcal{U}_L^T) = \prod_{\mathfrak{l}\in \mathrm{Ram}(L/F)\setminus\mathcal{T}} e_{\mathfrak{l}} = \prod_{\mathfrak{p}\in \mathrm{Ram}(L/k)_s\setminus T} e_{\mathfrak{p}}^2 \times \prod_{\mathfrak{p}\in \mathrm{Ram}(L/k)_i\setminus T} e_{\mathfrak{p}} \times \prod_{\mathfrak{p}\in \mathrm{Ram}(L/k)_r\setminus T} \frac{e_{\mathfrak{p}}}{2}. \tag{2.14}$$

where $\mathrm{Ram}(L/k)_s$ (*resp.* $\mathrm{Ram}(L/k)_i$, $\mathrm{Ram}(L/k)_r$) denotes the subset of $\mathrm{Ram}(L/k)$ consisting of primes that split (*resp.* that are inert, that ramify) in F/k.

Consider now to the right-hand side: when $i = 1$, the same argument as above shows that

$$\widehat{H}^1(D,\mathcal{U}_L^T) = \prod_{\mathfrak{p}\in \mathrm{Ram}(L/k)\setminus T} \widehat{H}^1(D(\mathfrak{P}),\mathcal{O}_{\mathfrak{P}}^{\times}).$$

and therefore

$$h^1(D,\mathcal{U}_L^T) = \prod_{\mathfrak{p}\in \mathrm{Ram}(L/k)\setminus T} e_{\mathfrak{p}} = \prod_{\mathfrak{p}\in \mathrm{Ram}(L/k)_s\setminus T} e_{\mathfrak{p}} \times \prod_{\mathfrak{p}\in \mathrm{Ram}(L/k)_i\setminus T} e_{\mathfrak{p}} \times \prod_{\mathfrak{p}\in \mathrm{Ram}(L/k)_r\setminus T} e_{\mathfrak{p}}. \tag{2.15}$$

When $i = 0$ we need to distinguish between places in T and not in T: using again that local units have trivial cohomology at unramified places together with the main statement of local class field theory ([Ser67, §2, Theorem 1]), we find

$$\widehat{H}^0(D,\mathcal{U}_L^T) = \prod_{\mathfrak{p}\notin T} \widehat{H}^0(D(\mathfrak{P}),\mathcal{O}_{\mathfrak{P}}^{\times}) \times \prod_{\mathfrak{p}\in T} \widehat{H}^0(D(\mathfrak{P}),L_{\mathfrak{P}}^{\times}) = \prod_{\mathfrak{p}\in \mathrm{Ram}(L/k)\setminus T} \widehat{H}^0(D(\mathfrak{P}),\mathcal{O}_{\mathfrak{P}}^{\times}) \times \prod_{\mathfrak{p}\in T} D(\mathfrak{P})^{\mathrm{ab}}.$$

Consider now a prime $\mathfrak{p}\in T$: if it lies in T_s, the decomposition subgroup $D(\mathfrak{P})$ is cyclic of odd order (dividing q), and $D(\mathfrak{P})^{\mathrm{ab}} \cong \mathbb{Z}/d_{\mathfrak{p}}$; when $\mathfrak{p}\notin T_s$, the decomposition group is either cyclic of order 2 or dihedral, and in both cases $D(\mathfrak{P})^{\mathrm{ab}} \cong \mathbb{Z}/2$. Combining this argument with [CN20, Lemma 3.9] we find

$$\widehat{h}^0(D,\mathcal{U}_L^T) = \prod_{\mathfrak{p}\in \mathrm{Ram}(L/k)_r\setminus T} 2 \cdot \prod_{\mathfrak{p}\in \mathrm{Ram}(L/k)_s\setminus T} e_{\mathfrak{p}} \cdot \prod_{\mathfrak{p}\in T\setminus T_s} 2 \cdot \prod_{\mathfrak{p}\in T_s} d_{\mathfrak{p}}. \tag{2.16}$$

Comparing (2.14) with (2.15) and (2.16) gives

$$h^1(G,\mathcal{U}_L^T) = h^1(D,\mathcal{U}_L^T) \cdot \widehat{h}^0(D,\mathcal{U}_L^T) \cdot 4^{|\mathrm{Ram}(L/k)_r|} \cdot 2^{|T\setminus T_s|} \cdot \prod_{\mathfrak{p}\in T_s} d_{\mathfrak{p}} \tag{2.17}$$

and after inverting 2 in (2.17) (taking into account that $d_{\mathfrak{p}}$ is odd for $\mathfrak{p}\in T_s$) we obtain the proposition. □

We are now ready to state the main result of this section:

Theorem 2.12. *With the notation introduced in Notation 2.5, the following equality holds:*

$$\frac{|Cl_L^T|\cdot|Cl_k^T|^2}{|Cl_F^T|\cdot|Cl_K^T|^2} = \prod_{\mathfrak{p}\in T_s} d_{\mathfrak{p}} \cdot \frac{\widehat{h}^0\big(D,E_L^T[\frac{1}{2}]\big)}{\widehat{h}^{-1}\big(D,E_L^T[\frac{1}{2}]\big)}$$

Proof. We proceed by first proving that the 2-parts of the left- and of the right-hand side coincide, and we eventually analyse the prime-to-2 parts. The cohomology groups appearing in the right-hand side have trivial 2-part, because $E_L^T[\frac{1}{2}]$ is a 2-divisible D-module; and all $d_{\mathfrak{p}}$ appearing in the product are odd, as observed in the proof of Proposition 2.11, after (2.17). Hence the 2-adic valuation of the right-hand side is trivial, as is the one of the left-hand side by Lemma 2.9.

We now move on to prove the equality for the prime-to-2 parts, replacing all T-class groups $Cl_\bullet^T$ by their prime-to-2 components $Cl_\bullet^T[\frac{1}{2}]$ in what follows. Applying Proposition 2.6, we just need to verify that

$$\frac{\big|Cl_k^T[\frac{1}{2}]\big|^2}{\big|Cl_F^T[\frac{1}{2}]\big|} \stackrel{?}{=} \prod_{\mathfrak{p}\in T_s} d_{\mathfrak{p}} \cdot \frac{\widehat{h}^0\big(D,E_L^T[\frac{1}{2}]\big)}{\widehat{h}^{-1}\big(D,E_L^T[\frac{1}{2}]\big)} \cdot \frac{h^0(D,Cl_L^T[\frac{1}{2}])\cdot\widehat{h}^{-1}(D,Cl_L^T[\frac{1}{2}])\cdot|\mathrm{Nm}_D Cl_L^T[\frac{1}{2}]|}{h^0(G,Cl_L^T[\frac{1}{2}])}. \tag{2.18}$$

Invoking the first equality of Lemma 2.10 twice (once for $H = G$ and once for $H = D$), the left-hand side of (2.18) satisfies the equality

$$\frac{\left|Cl_k^T[\frac{1}{2}]\right|^2}{\left|Cl_F^T[\frac{1}{2}]\right|} = \frac{h^1(G,\mathcal{U}_L^T[\frac{1}{2}])\cdot h^1(G,\mathrm{Pr}_L^T[\frac{1}{2}])}{h^0(G,Cl_L^T[\frac{1}{2}])\cdot h^1(G,E_L^T[\frac{1}{2}])}\cdot\frac{h^0(D,Cl_L^T[\frac{1}{2}])\cdot h^1(D,E_L^T[\frac{1}{2}])}{h^1(D,\mathcal{U}_L^T[\frac{1}{2}])\cdot h^1(D,\mathrm{Pr}_L^T[\frac{1}{2}])}\cdot\left|Cl_k^T[\tfrac{1}{2}]\right|;$$

the right-hand side further expands, through the second equality established in Lemma 2.10, to

$$\frac{h^1(G,\mathcal{U}_L^T[\frac{1}{2}])\cdot h^1(G,\mathrm{Pr}_L^T[\frac{1}{2}])}{h^0(G,Cl_L^T[\frac{1}{2}])\cdot h^1(G,E_L^T[\frac{1}{2}])}\cdot\frac{h^0(D,Cl_L^T[\frac{1}{2}])\cdot h^1(D,E_L^T[\frac{1}{2}])}{h^1(D,\mathcal{U}_L^T[\frac{1}{2}])\cdot h^1(D,\mathrm{Pr}_L^T[\frac{1}{2}])}\cdot\frac{\widehat{h}^{-1}(D,Cl_L^T[\frac{1}{2}])\cdot\left|\mathrm{Nm}_D Cl_L^T[\frac{1}{2}]\right|\cdot\widehat{h}^0(D,E_L^T[\frac{1}{2}])}{\widehat{h}^0(D,\mathcal{U}_L^T[\frac{1}{2}])\cdot\widehat{h}^{-1}(D,\mathrm{Pr}_L^T[\frac{1}{2}])}.$$

Upon reordering, the left-hand side in (2.18) is thus equal to

$$\frac{\left|Cl_k^T[\frac{1}{2}]\right|^2}{\left|Cl_F^T[\frac{1}{2}]\right|} = \frac{\widehat{h}^0\big(D,E_L^T[\frac{1}{2}]\big)\cdot h^0(D,Cl_L^T[\frac{1}{2}])}{h^0(G,Cl_L^T[\frac{1}{2}])}\cdot\frac{h^1(D,E_L^T[\frac{1}{2}])}{h^1(G,E_L^T[\frac{1}{2}])}\cdot\frac{h^1(G,\mathcal{U}_L^T[\frac{1}{2}])}{h^1(D,\mathcal{U}_L^T[\frac{1}{2}])\cdot\widehat{h}^0(D,\mathcal{U}_L^T[\frac{1}{2}])}\cdot$$

$$\cdot\frac{h^1(G,\mathrm{Pr}_L^T[\frac{1}{2}])}{h^1(D,\mathrm{Pr}_L^T[\frac{1}{2}])\cdot\widehat{h}^{-1}(D,\mathrm{Pr}_L^T[\frac{1}{2}])}\cdot\widehat{h}^{-1}(D,Cl_L^T[\tfrac{1}{2}])\cdot|\mathrm{Nm}_D Cl_L^T[\tfrac{1}{2}]|.$$

The third factor of the right-hand side is equal to $\prod_{\mathfrak{p}\in T_s} d_{\mathfrak{p}}$, by Proposition 2.11. Furthermore, [CN20, Proposition 2.1] implies that the fourth factor is equal to 1 whereas the second equals $\widehat{h}^{-1}(D,E_L^T[\frac{1}{2}])^{-1}$: hence the above equality simplifies to

$$\frac{\left|Cl_k^T[\frac{1}{2}]\right|^2}{\left|Cl_F^T[\frac{1}{2}]\right|} = \frac{\widehat{h}^0\big(D,E_L^T[\frac{1}{2}]\big)\cdot h^0(D,Cl_L^T[\frac{1}{2}])}{h^0(G,Cl_L^T[\frac{1}{2}])}\cdot\frac{\prod_{\mathfrak{p}\in T_s} d_{\mathfrak{p}}}{\widehat{h}^{-1}(D,E_L^T[\frac{1}{2}])}\cdot\widehat{h}^{-1}(D,Cl_L^T[\tfrac{1}{2}])\cdot|\mathrm{Nm}_D Cl_L^T[\tfrac{1}{2}]|$$

which is precisely (2.18). This concludes the proof of the theorem. □

Corollary 2.13. *In the same setting as in Theorem 2.12,*

$$\frac{|Cl_L^T|\cdot|Cl_k^T|^2}{|Cl_F^T|\cdot|Cl_K^T|^2} = \prod_{\mathfrak{p}\in T_s} d_{\mathfrak{p}}\cdot\frac{\widehat{h}^0(D,E_L^T)}{\widehat{h}^{-1}(D,E_L^T)}\cdot\frac{\widehat{h}^{-1}(\Sigma,E_L^T)}{\widehat{h}^0(\Sigma,E_L^T)}.$$

Proof. Observe that, for all $i\in\mathbb{Z}$, there is an isomorphism

$$\widehat{H}^i(D,E_L^T[\tfrac{1}{2}]) \cong \widehat{H}^i(D,E_L^T)/\widehat{H}^i(D,E_L^T)^{(2)}.$$

The corollary follows by combining Theorem 2.12 with [CN20, Lemma 2.9]. □

3. ℓ-class groups in fake $\mathbb{Z}_p$-extensions of dihedral type

Fix a number field F, we let $p\geq 3$ be an odd prime and we let L_∞/F be a $\mathbb{Z}_p$-extension, with layers L_n such that $[L_n : F] = p^n$ (so that $L_0 = F$). The goal of this section is to study the ℓ-adic behaviour (for $\ell\neq p$) of T-class groups in fake $\mathbb{Z}_p$-extensions. We begin with the definition of this concept, taken from [CN23, Definition 4.1]:

Definition 3.1. Suppose that there exists a subfield $k\subseteq F$ such that, for every $n\geq 0$, L_n/k is Galois, with dihedral Galois group $\mathrm{Gal}(L_n/k)\cong D_{p^n}$ of order $2p^n$. Denote by K_n a subfield of index 2 in L_n/k, chosen so that $K_n\supseteq K_{n-1}$ for all $n\geq 1$, and put $K_\infty = \bigcup K_n$. The extension K_∞/k is said to be a fake $\mathbb{Z}_p$-extensions of dihedral type, the extension L_∞/k is the Galois closure of the fake $\mathbb{Z}_p$-extension and the field F is said to be the normalizing quadratic extension of K_∞/k.

Fix from now on a fake $\mathbb{Z}_p$-extension of dihedral type K_∞/k, supposing that F/k is the normalizing quadratic extension. Fix also a finite set of primes T of k and let ℓ be an odd prime different from p.

Remark 3.2. The tower of extensions $K_\infty = \bigcup K_n/k$ bears strong similarities with a $\mathbb{Z}_p$-extension, with the crucial difference that its layers are all *non-Galois* extensions. In particular, the usual approach of obtaining useful information by analysing Galois actions (pioneered by Iwasawa in [Iwa73] for the p-adic behaviour, and recently developed by Bandini–Longhi in [BL23] for the ℓ-adic counterpart, where $\ell \neq p$) breaks down completely for the fake $\mathbb{Z}_p$-extension. In [CN23] we developed other techniques, based on the analysis of Tate cohomology of normic systems, in order to address these questions in the non-Galois setting.

Following [BL23, § 4.2], we define the following sequence of natural numbers (for $n \geq 0$):

$$f_n(\ell, p) = f_n = \text{order of } \ell \text{ in } (\mathbb{Z}/p^n)^\times.$$

As remarked *ibid.*, setting $a = v_p(\ell^{f_1} - 1)$, it is easy to check that these values satisfy

$$f_n = [\mathbb{Q}_\ell(\mu_{p^n}) : \mathbb{Q}_\ell] = \begin{cases} 1 & \text{if } n = 0 \\ f_1 & \text{if } 1 \leq n \leq a \\ p^{n-a} f_1 & \text{if } n > a \end{cases} \tag{3.1}$$

where μ_{p^n} denotes the group of p^n-th roots of unity.

To define a second sequence of natural numbers that occur in the statement of Theorem 3.5 below we need some preliminaries.

Notation 3.3. For any field $\bullet \in \{k, F, L_n, K_n\}_{n\geq 0}$ let $A_\bullet^T$ be the ℓ-Sylow $(Cl_\bullet^T)^{(\ell)}$ of the T-class group of $\bullet$. Denote by $e_{[n]}$, for $n \geq 0$, the idempotents in the topological $\mathbb{Z}_\ell$-algebra $\Lambda = \mathbb{Z}_\ell[\![\mathrm{Gal}(L_\infty/F)]\!]$ defined in [BL23, Theorem 2.1]; letting $X = \varprojlim A_{L_n}^T$ (where the transition morphisms are induced by the norm), write $X_{[n]}$ for the projection $X_{[n]} = e_{[n]}X$. We set

$$t_{[n]}^{\mathrm{Iw}} = \frac{v_\ell(|X_{[n]}|)}{f_n} \quad (\text{for } n \geq 0) \qquad \text{and} \qquad t_{[n]}^{\mathrm{fk}} = \begin{cases} \dfrac{v_\ell\big(|A_{K_n}^T/A_{K_{n-1}}^T|\big)}{f_n} & \text{for } n \geq 1 \\ \dfrac{t_{[0]}^{\mathrm{Iw}}}{2} & \text{for } n = 0. \end{cases}$$

Remark 3.4. Clearly, the above values satisfy $f_m t_{[m]}^{\mathrm{fk}} \in \mathbb{Z}$ for $m \geq 1$ and $f_0 t_{[0]}^{\mathrm{fk}} \in \mathbb{Z}[\frac{1}{2}]$.

Our main result is the following

Theorem 3.5. *In the setting of Notation 3.3, for all $n \geq 0$ the ℓ-adic valuation of the order $|A_{K_n}^T|$ satisfies*

$$v_\ell\big(|A_{K_n}^T|\big) = \nu + \sum_{m=0}^{n} f_m t_{[m]}^{\mathrm{fk}} = \nu + t_{[0]}^{\mathrm{fk}} + f_1\Big(\sum_{m=1}^{a} t_{[m]}^{\mathrm{fk}} + \sum_{m=a+1}^{n} p^{m-a} t_{[m]}^{\mathrm{fk}}\Big).$$

where $a = v_p(\ell^{f_1} - 1)$ and where $\nu \in \mathbb{Z}[\frac{1}{2}]$ is a constant independent of n.

Proof. The second equality follows from the first via the relations in (3.1). Concerning the first equality, [BL23, Theorem 4.6] shows[3] that, for all $n \geq 0$,

$$v_\ell\big(|A_{L_n}^T|\big) = \sum_{m=0}^{n} f_m t_{[m]}^{\mathrm{Iw}}. \tag{3.2}$$

The key ingredient is our Theorem 2.12: since for every prime $\mathfrak{p}$ in T_s, the local degree $\mathfrak{p}$ is a power of p, the ℓ-adic valuation $v_\ell(d_\mathfrak{p})$ is trivial and Theorem 2.12 implies that for all $n \geq 0$,

$$\begin{aligned} v_\ell\big(|A_{L_n}^T|\big) &= 2 \cdot v_\ell\big(|A_{K_n}^T|\big) + v_\ell\big(|A_F^T|\big) - 2 \cdot v_\ell\big(|A_k^T|\big) + v_\ell\big(\widehat{h}^0(D_n, \mathcal{O}_{L_n}^\times)\big) - v_\ell\big(\widehat{h}^{-1}(D_n, \mathcal{O}_{L_n}^\times)\big) \\ &= 2v_\ell\big(|A_{K_n}^T|\big) - 2\nu \end{aligned} \tag{3.3}$$

[3] The main result in *loc. cit.* is proved for $T = \emptyset$ but, as observed in [BL23, §5], the proof relies on general algebraic properties that extend straightforwardly to T-class groups for any finite T.

where $\nu = v_\ell(|A_k^T|) - v_\ell(|A_F^T|)/2$ is independent of n; and where the two terms $\widehat{h}^i(D_n, \mathcal{O}_{L_n}^\times)$ have trivial ℓ-adic valuation since they are orders of cohomology groups of D_n and thus are only divisible by 2 and p. Moreover, [BL23, Theorem 4.3] yields isomorphisms[4] $X_{[m+1]} \cong A_{L_{m+1}}^T / A_{L_m}^T$ for all $m \geq 0$, and $X_{[0]} \cong A_F^T$; hence, for all $m \geq 0$,

$$\begin{aligned} t_{[m+1]}^{\mathrm{Iw}} &= \frac{v_\ell\big(|X_{[m+1]}|\big)}{f_{m+1}} \\ &= \frac{v_\ell\big(|A_{L_{m+1}}^T|\big) - v_\ell\big(|A_{L_m}^T|\big)}{f_{m+1}} \\ &= \frac{2v_\ell\big(|A_{K_{m+1}}^T|\big) - 2\nu - 2v_\ell\big(|A_{K_m}^T|\big) + 2\nu}{f_{m+1}} \\ &= \frac{2v_\ell\big(|A_{K_{m+1}}^T / A_{K_m}^T|\big)}{f_{m+1}} \\ &= 2t_{[m+1]}^{\mathrm{fk}} \end{aligned} \tag{3.4}$$

Combining (3.3) with (3.2) and (3.4) (and recalling that, by definition, $t_{[0]}^{\mathrm{Iw}} = 2t_{[0]}^{\mathrm{fk}}$) one obtains that, for all $n \geq 0$,

$$\begin{aligned} 2v_\ell\big(|A_{K_n}^T|\big) &= v_\ell\big(|A_{L_n}^T|\big) + 2\nu \\ &= 2\nu + t_{[0]}^{\mathrm{Iw}} + \sum_{m\geq 0} f_{m+1} t_{[m+1]}^{\mathrm{Iw}} \\ &= 2\nu + 2t_{[0]}^{\mathrm{fk}} + \sum_{m\geq 0} 2f_{m+1} t_{[m+1]}^{\mathrm{fk}}. \end{aligned}$$

Dividing both members of the above equalities by 2 finishes the proof of the theorem. □

Having established Theorem 3.5, that is the analogue of [BL23, Theorem 4.6] for the fake $\mathbb{Z}_p$-extension, we can obtain the following corollary, that generalises [BL23, Theorem 4.8] (we only state it for $T = \emptyset$, for simplicity):

Corollary 3.6. *For every odd prime q, the sequence $|Cl_{K_n}{}^{(q)}|$ of the orders of the q-Sylow subgroups of Cl_{K_n} converges p-adically.*

Proof. For $q = p$ this follows from the main formula for the growth of p-Sylow subgroups established in [CN23, Theorem 4.6]. When $q \neq p$, we can apply Theorem 3.5 with $\ell = q$ (so $Cl_{K_n}{}^{(q)} = A_{L_n} = A_{L_n}^{\emptyset}$) to find that for all $n \geq a$,

$$\Big|A_{K_{n+1}}\Big| - \big|A_{K_n}\big| = \big|A_{K_n}\big|\Big(\ell^{f_1 p^{n+1-a} t_{[n+1]}^{\mathrm{fk}}} - 1\Big) = \big|A_{K_n}\big|\Big(\ell^{f_{n+1} t_{[n+1]}^{\mathrm{fk}}} - 1\Big). \tag{3.5}$$

As observed in Remark (3.4), the term $f_{n+1} t_{[n+1]}^{\mathrm{fk}}$ is an integer. Since $\ell^{f_{n+1}} \equiv 1 \pmod{p^{n+1}}$ by definition, it follows that the right-hand side of (3.5) tends to 0 p-adically: therefore, the sequence of $|A_{K_n}|$ is Cauchy and its limit exists, establishing the corollary. □

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